TWISTS FOR COHOMOLOGICAL HALL ALGEBRAS

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Abstract. We make a short review of twists in the theory of cohomological Hall algebras.

Contents

In the theory of cohomological Hall algebras (CoHAs) for quivers with potentials (Q, W) or preprojective algebras of quivers, various twists appear. There are two different origins for the twists. The first twists originate from the desire to identify the CoHAs with Yangian-type quantum groups. More precisely, the CoHA of quiver with potential is expected to be a deformation of the enveloping algebra $\mathbf{U}(\mathfrak{g}_{(Q,W)}[u])$ of the loop BPS Lie algebra $\mathfrak{g}_{(Q,W)}[u]$ of the quiver with potential. However, before twist, the CoHA is rather isomorphic to the specialization at -1 of the quantum deformation $\mathbf{U}_{-1}(\mathfrak{g}[u])$. The same is true for the CoHA of the preprojective algebra of a quiver. The second source if twists are the cohomological dimensional reduction isomorphisms.

1. Twists

Let A be an algebra graded by a monoid M. We denote by $m\colon A\otimes A\to A$ the multiplication. The restriction of m to $A_d\otimes A_e$ is denoted by $m_{d,e}$ for $d,e\in M$. Given a bilinear form $\psi\colon M\times M\to \mathbf{Z}/2\mathbf{Z}$, one can twist the multiplication by replacing $m_{d,e}$ by $(-1)^{\psi(d,e)}m_{d,e}$.

2. Enveloping algebra twist

2.1. **Quivers with potentials.** Let (Q, W) be a quiver with potential. The pullback-pushforward procedure in critical cohomology produces a associative algebra structure on $A_{Q,W} := \mathrm{H}^*(\mathfrak{M}_{Q,W}, \varphi_{\mathrm{Tr}(W)})$. This is an $\mathbf{N}^{Q_0}$-graded algebra. In order to identify it to an enveloping algebra, it is necessary to twist the multiplication by a bilinear form $\psi\colon \mathbf{N}^{Q_0}\times\mathbf{N}^{Q_0}\to\mathbf{Z}/2\mathbf{Z}$ such that

$$\psi(d,d')+\psi(d',d)=\tau(d,d')$$

where

$$\tau(d,d')=\chi_Q(d,d')+\chi_Q(d,d)\chi_Q(d',d')$$

and χ_Q is the Euler form of the quiver Q. Given such a choice of ψ, any translate of ψ by a *symmetric* bilinear form is again a valid choice. This means that τ could be defined by just the Euler form of Q.

Example 2.1. We let $\widetilde{Q}$ be the triple quiver associated to a given quiver Q. Then,

$$\chi_{\widetilde{Q}}(d,d')=\chi_Q(d,d')+\chi_Q(d',d)-d\cdot d'\,.$$

The Euler form of Q is a valid choice for ψ.

Date: February 19, 2025.

This twists appears in [KS11, §2.6], [Efi12, §2.3], and [DM20, (15)].

2.2. Preprojective algebras of quivers. Given a quiver Q, one can construct an algebra structure on the Borel–Moore homology $A_{\Pi_Q} \coloneqq \mathrm{H}^{\mathrm{BM}}_*(\mathfrak{M}_{\Pi_Q}, \mathbf{Q})$ of the stack of representations of Π_Q. It is a $\mathbf{N}^{Q_0}$-graded algebra. In order to make it an enveloping algebra (or satisfy a PBW theorem), one needs to twist the multiplication by a bilinear form $\phi\colon \mathbf{N}^{Q_0} \times \mathbf{N}^{Q_0} \to \mathbf{Z}/2\mathbf{Z}$ such that

$$\phi(d, d') + \phi(d', d) = \chi_Q(d, d') + \chi_Q(d', d)\,.$$

It is clear that the Euler form of Q itself is a valid choice. The twists are explained in [Hen24; DHM23].

3. Dimensional reduction twist

Let Q be a quiver. Then, the dimensional reduction isomorphisms provide an isomorphism

$$\mathsf{dr}\colon \mathrm{H}^{\mathrm{BM}}_*(\mathfrak{M}_{\Pi_Q}, \mathbf{Q}) \to \mathrm{H}^*(\mathfrak{M}_{(\widetilde{Q}, \widetilde{W})}, \mathbf{Q})$$

which is an isomorphism of algebras when the multiplication is twisted by $(d, d') \mapsto d \cdot d'$. This is proven in [RS17, Appendix by Ben Davison] Since $(d, d') \mapsto d \cdot d'$ is symmetric, it is easy to see that a valid choice of twist for $(\widetilde{Q}, \widetilde{W})$ is transferred via dimensional reduction to a valid twist for Π_Q.

References

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