The degree zero BPS Lie algebra of a curve

Lucien Hennecart

Séminaire de la Tortue

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joint work with Ben Davison & Olivier Schiffmann.

X smooth projective curve

Coherent sheaves on X

"1-dimensional"

perverse sheaves on Coh

categorified spherical Hall algebra

Higgs sheaves on X

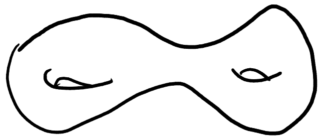
"2-dimensional"

BM homology of Higgs / global nilpotent cone

Cohomological Hall algebra

Characteristic cycle map

I - Context

$X =$  smooth projective curve $/\mathbb{C}$, genus g

[more generally, $X =$ weighted projective curve
$\qquad\qquad\qquad\quad \parallel$
1 dimensional Deligne-Mumford stack]

Coherent sheaves on X

$$\mathrm{Coh}(X) = \bigsqcup_{\alpha \in \mathbb{Z}^+} \mathrm{Coh}_\alpha(X)$$ stack of coherent sheaves on X,

$$\mathbb{Z}^+ = \{(r,d) \in \mathbb{Z}^2 \mid r > 0 \text{ or } (r = 0, d > 0)\}.$$

$\mathrm{Coh}_\alpha(X)$ is a smooth Artin stack locally of finite type

$$\dim \mathrm{Coh}_\alpha(X) = -\langle \alpha, \alpha \rangle$$

Euler form of X: $$\langle -,- \rangle : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$$
$$((r,d),(r',d')) \longmapsto (1-g)rr' + rd' - r'd$$

Higgs sheaves on X

$$\mathrm{Higgs}(X) = \bigsqcup_{\alpha \in \mathbb{Z}^+} \mathrm{Higgs}_\alpha(X)$$ Artin stack of Higgs sheaves on X.

Ω_X canonical bundle of X

$\Omega_{\mathbb{P}^1} \simeq \mathcal{O}(-2)$

$\Omega_E \simeq \mathcal{O}_E$ E elliptic curve

$\deg \Omega_X = 2g-2$ in general.

$$\mathrm{Higgs}(X)(\mathbb{C}) = \left\{ (\mathcal{F}, \theta) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ coherent sheaf on } X \\ \theta : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega_X \quad \mathcal{O}_X\text{-linear} \end{array} \right\} \Big/ \mathrm{iso}$$

$\mathrm{Higgs}(X)$ is locally of finite type

$\dim \mathrm{Higgs}(X) = -2\langle \alpha, \alpha \rangle$.

Locally, $\mathrm{Higgs}(X)$ can be constructed by stacky symplectic reduction using the fact that $\mathrm{Coh}(X)$ is locally a quotient stack.

In rough terms, $\mathrm{Higgs}_\alpha(X) = T^* \mathrm{Coh}_\alpha(X)$.

can give a precise sense to this.

Global nilpotent cone

$\Lambda = \bigsqcup_{\alpha \in \mathbb{Z}^+} \Lambda_\alpha \subset \mathrm{Higgs}(X)$ closed substack of nilpotent Higgs sheaves

$(\mathcal{F}, \theta)$ is nilpotent if $\exists s \geq 0$, $(\theta^s : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^s) = 0$.

Laumon, Ginzburg, Beilinson-Drinfeld : Λ is Lagrangian in $\mathrm{Higgs}(X)$.

Λ_α is reducible, not reduced : very singular.

example : $X = \mathbb{P}^1$: $\Lambda_\alpha = \mathrm{Higgs}_\alpha(X)$

$X = \mathbb{A}^1$: $\mathrm{Coh}_{(0,d)}(\mathbb{A}^1) \simeq \mathfrak{gl}_d / GL_d$

$\mathrm{Higgs}_{(0,d)}(\mathbb{A}^1) \simeq \mathrm{Comm}(\mathfrak{gl}_d) / GL_d$ commuting variety of $\mathfrak{gl}_d$

$$\Lambda_{(0,d)} \simeq \{(x,y) \in \mathfrak{gl}_d \times \mathcal{N}_d \mid [x,y] = 0\} / GL_d$$

where $\mathcal{N}_d \subset \mathfrak{gl}_d$ is the nilpotent cone.

$T^*\mathfrak{gl}_d \cong \mathfrak{gl}_d \times \mathfrak{gl}_d$ via trace pairing.

$$\begin{array}{c} T^*\mathfrak{gl}_d \\ \cup \\ \Lambda_{(0,d)} \end{array}$$

2 projections :

$$\begin{tikzcd} & (x,y) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ x & & y \\ & \Lambda_{(0,d)} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[d, phantom, "\cap"] & \\ \mathfrak{gl}_d & \mathfrak{gl}_d \times \mathfrak{gl}_d & \mathcal{N}_d \subset \mathfrak{gl}_d \end{tikzcd}$$

With respect to p_2,

$$\Lambda_{(0,d)} = \bigsqcup_{\substack{G \subset \mathcal{N}_d \\ \text{nilp. orbit}}} T^*_G \mathfrak{gl}_d .$$

With respect to p_1,

$$\Lambda_{(0,d)} = \bigsqcup_{\lambda \in \mathcal{P}_d} \overline{T^*_{\boxminus(\lambda)} \mathfrak{gl}_d} \qquad \mathcal{P}_d = \{ \text{partitions of } d \}$$

$$\boxminus(\lambda) = \left\{ x \in \mathfrak{gl}_d \;\middle|\; x \text{ is in the orbit of } \begin{pmatrix} J_{\lambda_1}(x_1) & & 0 \\ & \ddots & \\ 0 & & J_{\lambda_r}(x_r) \end{pmatrix} \right\}$$

if $\lambda = (\lambda_1, \dots, \lambda_r)$, for some x_i's pairwise distinct.

& $J_\lambda(x) = \begin{pmatrix} x & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & x \end{pmatrix}$

$\lambda \in \mathbb{N}, x \in \mathbb{C}$

II - Spherical Eisenstein perverse sheaves on Coh(X)

Induction / Restriction diagram

$\alpha, \beta \in \mathbb{Z}^+$

$$\begin{tikzcd} & \widetilde{Coh}_{\alpha,\beta} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Coh_\alpha \times Coh_\beta & & Coh_{\alpha+\beta} \end{tikzcd}$$

$$\widetilde{Coh}_{\alpha,\beta}(\mathbb{C}) = \left\{ (\mathcal{G} \subset \mathcal{F}) \;\middle|\; \begin{array}{l} [\mathcal{F}] = \alpha+\beta \\ [\mathcal{G}] = \beta \end{array} \right\} / \sim$$

$$p(\mathcal{G} \subset \mathcal{F}) = \mathcal{F}$$

$$q(\mathcal{G} \subset \mathcal{F}) = (\mathcal{F}/\mathcal{G}, \mathcal{G})$$

<u>Facts</u> : p is proper

q is smooth of relative dimension $-\langle \alpha, \beta \rangle$.

Induction functor

$$\mathrm{Ind}_{\alpha,\beta} : D^b_c(\mathrm{Coh}_\alpha) \times D^b_c(\mathrm{Coh}_\beta) \longrightarrow D^b_c(\mathrm{Coh}_{\alpha+\beta})$$

$$(\mathcal{F}, \mathcal{G}) \longmapsto p_* q^* \mathcal{F} \boxtimes \mathcal{G}[-\langle\alpha,\beta\rangle]$$

$\rightsquigarrow$ associative multiplication on

$$K_0(D^b_c(\mathrm{Coh})) := \bigoplus_{\alpha \in \mathbb{Z}^+} K_0(D^b_c(\mathrm{Coh}_\alpha)).$$

$\rightsquigarrow$ $\mathrm{Ind}_{\alpha,\beta}$ preserves semisimple complexes.

$D^b_c(\mathrm{Coh}_\alpha)$ is too big; need to select a subcategory.

$\mathcal{Q}$ = smallest triangulated category of $D^b_c(\mathrm{Coh})$ stable under direct summands, $\mathrm{Ind}_{\alpha,\beta}$ $(\alpha, \beta \in \mathbb{Z}^+)$ and containing $\underline{\mathbb{C}}_{\mathrm{Coh}_\alpha}$ for $\alpha \in \mathbb{Z}^+$.

$\mathcal{P}$ = full subcategory of perverse sheaves in $\mathcal{Q}$.
elements of $\mathcal{P}$ are semisimple (thanks to the decomposition theorem and the properness of p).

Restriction functor

$$\begin{aligned} \mathrm{Res}_{\alpha,\beta} : D^b_c(\mathrm{Coh}_{\alpha+\beta}) &\longrightarrow D^b_c(\mathrm{Coh}_\alpha \times \mathrm{Coh}_\beta) \\ \mathcal{F} &\longmapsto q_! p^* \mathcal{F}[-\langle \alpha,\beta\rangle] \end{aligned}$$

There is a way to understand $\mathrm{Res}_{\alpha,\beta}$ through hyperbolic localization (Braden, Drinfeld-Gaitsgory for stacks)

Fact: $\mathrm{Res}_{\alpha,\beta} : \mathcal{Q}_{\alpha+\beta} \longrightarrow \mathcal{Q}_\alpha \boxtimes \mathcal{Q}_\beta$.

The (undeformed) spherical Hall algebra

$(K_0(\mathcal{Q}), \mathrm{Ind}, \mathrm{Res})$ is a bialgebra.

1 **Conjecture:** It is cocommutative: $\mathrm{Res} = \mathrm{Res}^{op}$.

In other words, $\mathrm{Res}_{\alpha,\beta} \cong \mathrm{Tw} \circ \mathrm{Res}_{\beta,\alpha}$

$$\mathrm{Tw} : \quad K_0(\mathcal{Q}_\alpha) \times K_0(\mathcal{Q}_\beta) \longrightarrow K_0(\mathcal{Q}_\beta) \times K_0(\mathcal{Q}_\alpha).$$

2 **Conjecture**: Elements of $\mathcal{P}$ are Verdier self-dual

2 $\Rightarrow$ 1 thanks to hyperbolic localization

III - The cohomological Hall algebras of X

$$\mathrm{CoHA}(X) = \bigoplus_{\alpha \in \mathbb{Z}^+} H_*^{BM}\left(\mathrm{Higgs}_\alpha(X)\right)$$

$$\mathrm{CoHA}_\Lambda(X) = \bigoplus_{\alpha \in \mathbb{Z}^+} H_*^{BM}(\Lambda_\alpha)$$

+ algebra, coalgebra structures on these $\mathbb{Z}^+$-graded vector spaces.

Degree zero CoHA [algebra of interest to us]

$$\mathrm{CoHA}_\Lambda^{top}(X) := \bigoplus_{\alpha \in \mathbb{Z}^+} H_{top}^{BM}(\Lambda_\alpha)$$

$$H_{top}^{BM}(\Lambda_\alpha) = H^{BM}_{-\langle \alpha, \alpha \rangle}(\Lambda_\alpha) = \mathbb{C}[\mathrm{Irr}\,\Lambda_\alpha]$$

where $\mathrm{Irr}(\Lambda_\alpha) = \{$ irreducible components of $\Lambda_\alpha \}$

$\mathrm{CoHA}_\Lambda^{top}(X)$ is stable under the multiplication of $\mathrm{CoHA}_\Lambda(X)$:

$$\mathrm{CoHA}_\Lambda^{top}(X) \subset \mathrm{CoHA}_\Lambda(X) \subset \mathrm{CoHA}(X)$$ chain of algebra morphisms

Goal: understand as much as possible this algebra.

* explain how to define the product / coproduct of $\mathrm{CoHA}_\Lambda^{top}(X)$
* generation theorem for $\mathrm{CoHA}_\Lambda^{top}(X)$
* CC map
* Conjectures and Theorems

The multiplication of the CoHA

Induction diagram for Higgs sheaves

$\alpha, \beta \in \mathbb{Z}^+$

$$\begin{tikzcd} & \widetilde{\mathrm{Higgs}}_{\alpha,\beta} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Higgs}_\alpha \times \mathrm{Higgs}_\beta & & \mathrm{Higgs}_{\alpha+\beta}. \end{tikzcd}$$

$$\widetilde{\mathrm{Higgs}}_{\alpha,\beta} = \left\{ (\mathcal{G} \subset \mathcal{F}, \theta) \;\middle|\; \begin{array}{l} (\mathcal{F}, \theta) \text{ Higgs sheaf, } [\mathcal{F}] = \alpha+\beta \\ [\mathcal{G}] = \beta, \quad \theta(\mathcal{G}) \subset \mathcal{G} \otimes \Omega_X \end{array} \right\}$$

$$p((\mathcal{G} \subset \mathcal{F}, \theta)) = \mathcal{F}$$

$$q((\mathcal{G} \subset \mathcal{F}, \theta)) = \left((\mathcal{F}/\mathcal{G}, \theta|_{\mathcal{F}/\mathcal{G}}), (\mathcal{G}, \theta|_{\mathcal{G}}) \right).$$

- p is proper.
- q is not smooth, not lci: badly behaved.

But we can construct a virtual pullback

$$H^*_{BM}(\mathrm{Higgs}_\alpha \times \mathrm{Higgs}_\beta) \xrightarrow{q^!} H^*_{BM}(\widetilde{\mathrm{Higgs}}_{\alpha,\beta}).$$

strategy: • embed q locally in a l.c.i. morphism.
(Sala-Schiffmann)
• define the multiplication locally
• check that it glues. (painful)

Local multiplication

Local induction diagram

$\mathcal{L}, \mathcal{L}'$ line bundles $/X$ $\qquad$ $\deg \mathcal{L} \ll \deg \mathcal{L}' \ll 0$

$$\begin{tikzcd} \mathrm{Coh}_\alpha^{>\mathcal{L}} \times \mathrm{Coh}_\beta^{>\mathcal{L}} & \widetilde{\mathrm{Coh}}_{\alpha,\beta}^{>\mathcal{L},>\mathcal{L}'} \arrow[l, "q"'] \arrow[r] & \mathrm{Coh}_{\alpha+\beta}^{>\mathcal{L}'} \end{tikzcd}$$

$\mathcal{F}$ strongly generated by $\mathcal{L}$

$\mathcal{G} \subset \mathcal{F}$

$\mathcal{F}$ strongly generated by $\mathcal{L}$

$\mathcal{G}, \mathcal{F}/\mathcal{G}$ by $\mathcal{L}'$

All stacks above are quotient stacks: we have explicit algebraic varieties W°, X, X' and a group G such that

$$X/G \simeq \mathrm{Coh}_\alpha^{>\mathcal{L}} \times \mathrm{Coh}_\beta^{>\mathcal{L}} \times BU$$

$$W^\circ/G \simeq \widetilde{\mathrm{Coh}}_{\alpha,\beta}^{>\mathcal{L},\mathcal{L}'}$$

$$X'/G \simeq \mathrm{Coh}_{\alpha+\beta}^{>\mathcal{L}'}$$

$BU = pt/U$

U unipotent group.

and
$$\begin{tikzcd}
 & W^\circ \arrow[dl, "q"'] \arrow[dr, "p"] & \\
X & & X'
\end{tikzcd}
\quad . \qquad p\times q : W^\circ \xhookrightarrow[\text{closed}]{} X\times X' .$$

$$\begin{tikzcd}
 & Z^\circ_G \arrow[dl, "\phi_G"'] \arrow[dr, "\psi_G"] \arrow[d, hook] & \\
\mu^{-1}(0) = T^*_G X \arrow[dd, hook] & & T^*_G X' \arrow[dd, hook] \\
 & Z^\circ := T^*_{W^\circ}(X\times X') \arrow[dl, "\phi"', "\text{l.c.i morphism}"] \arrow[dr, "\psi"] & \\
T^*X & & T^*X'
\end{tikzcd}$$

$$T^*_G X/_G \simeq \mathrm{Higgs}_\alpha^{>\mathcal{L}} \times \mathrm{Higgs}_\beta^{>\mathcal{L}} \times \mathrm{BU}$$ no effect on the cohomology.

$$Z^\circ_G/_G \simeq \mathrm{Higgs}_{\alpha,\beta}^{\sim\,>\mathcal{L},\,>\mathcal{L}'}$$

$$X'/G \simeq \mathrm{Higgs}_{\alpha+\beta}^{>\mathcal{L}'}.$$

$$(\psi_G)_* \circ \phi_G^! : H_*^{BM,G}(T^*_G X) \longrightarrow H_*^{BM,G}(T^*_G X')$$

local multiplication

The comultiplication of the CoHA (sketch)

$$\mathrm{CoHA}(X) = \bigoplus_{\alpha \in \mathbb{Z}^+} H_*^{BM}(\mathrm{Higgs}_\alpha(X))$$

"infinitesimal inertia stack of Higgs" : classifies pairs $(\underset{\text{Higgs sheaf on } X}{\mathcal{F}}, \underset{\in \mathrm{End}(\mathcal{F})}{g})$

$\rightsquigarrow \overline{\mathrm{Higgs}(X)}$

dimensional reduction : $H_*^{BM}(\mathrm{Higgs}_\alpha) \cong H_*^{BM}(\overline{\mathrm{Higgs}_\alpha(X)}, \varphi_\alpha)$ "vanishing cycle coh."

$$\begin{tikzcd}
\widetilde{\mathrm{Higgs}_\alpha}(X) \arrow[r, "\tilde{\Delta}"] \arrow[d] & \overline{\mathrm{Higgs}_\alpha}(X) \arrow[d, "\text{eigenvalues of } g"] \\
H^0(X, \mathcal{O}_X) \arrow[r, "\text{diagonal}"'] & H^0(X, \mathcal{O}_X)^{\oplus r} / \mathfrak{S}_r
\end{tikzcd}$$

$\tilde{\Delta}^*$ is an iso in cohomology

$$\begin{tikzcd}
\widetilde{\widetilde{\mathrm{Higgs}_\alpha}}(X) \arrow[r, "\tilde{\tilde{\Delta}}"] \arrow[d] & \overline{\mathrm{Higgs}_\alpha}(X) \arrow[d, "\text{eigenvalues of } g"] \\
H^0(X, \mathcal{O}_X)^{\oplus 2} / \mathfrak{S}_2 \arrow[r] & H^0(X, \mathcal{O}_X)^{\oplus r} / \mathfrak{S}_r \\
(f, g) \arrow[r, mapsto] & (\underbrace{f, \dots, f}_{r_1}, \underbrace{g, \dots, g}_{r_2})
\end{tikzcd}$$

Thom-Sebastiani: $H_*^{BM}(\widetilde{\mathrm{Higgs}}_\alpha(X)) \simeq \bigoplus_{(r_1, d_1) + (r_2, d_2) = \alpha} H_*^{BM}(\mathrm{Higgs}_{(r_1, d_1)}) \otimes H_*^{BM}(\mathrm{Higgs}_{(r_2, d_2)})$

Combining $(\tilde{\Delta}^*)^{-1}$; $\tilde{\tilde{\Delta}}^*$ and dimensional reduction, we obtain a map

$$H_*^{BM}(\mathrm{Higgs}_\alpha) \longrightarrow \bigoplus_{\alpha_1 + \alpha_2 = \alpha} H_*^{BM}(\mathrm{Higgs}_{\alpha_1}) \otimes H_*^{BM}(\mathrm{Higgs}_{\alpha_2})$$

The characteristic cycle map – general definition/properties

X smooth $\mathbb{C}$-variety

$D^b_c(X)$ category of constructible complexes on X

$\mathbb{Z}[\mathrm{Lagr}^{\mathbb{C}^*}(T^*\mathbb{C})]$ – functions $\mathrm{Lagr}^{\mathbb{C}^*}(T^*X) \longrightarrow \mathbb{Z}$ with finite support.

$\mathrm{Lagr}^{\mathbb{C}^*}$ // closed, conical, Lagrangian subvarieties of T^*X

$$CC : K_0(D^b_c(X)) \longrightarrow \mathbb{Z}[\mathrm{Lagr}^{\mathbb{C}^*}(T^*X)]$$

- morphism of abelian groups
- functoriality w.r.t. smooth pull-backs and proper push-forwards.
- normalization : $CC(\mathcal{L}) = [T^*_X X]$

$\mathcal{L}$ local system

- Functoriality:
  * If $Y \xrightarrow{f} X$ smooth,

$$\begin{tikzcd}
K_0(D^b_c(X)) \arrow[r] \arrow[d, "f^*"'] \arrow[dr, phantom, "G"] & \mathbb{Z}[\mathrm{Lagr}^{\mathbb{C}^*}(T^*X)] \arrow[d, "f^*"] \\
K_0(D^b_c(Y)) \arrow[r] & \mathbb{Z}[\mathrm{Lagr}^{\mathbb{C}^*}(T^*Y)]
\end{tikzcd}$$

  * If $Y \to X$ proper,

$$\begin{tikzcd}
K_0(D^b_c(Y)) \arrow[r] \arrow[d, "f_*"'] \arrow[dr, phantom, "G"] & \mathbb{Z}[\mathrm{Lagr}^{\mathbb{C}^*}(T^*Y)] \arrow[d, "f_*"] \\
K_0(D^b_c(X)) \arrow[r] & \mathbb{Z}[\mathrm{Lagr}^{\mathbb{C}^*}(T^*X)]
\end{tikzcd}$$

The link between $K_0(\mathcal{P})$ and $CoHA_\Lambda^{top}(X)$

The CC map induces a map

$$\boxed{CC : K_0(\mathcal{P}) \longrightarrow CoHA_\Lambda^{top}(X)}.$$

Need to check: If $\mathcal{F} \in \mathcal{P}$, $SS(\mathcal{F}) := supp(CC(\mathcal{F})) \subset \Lambda$.

<u>Proof</u>: $\exists\, \alpha, \alpha_1, \dots, \alpha_s \in \mathbb{Z}^+$ $\quad \alpha = \sum \alpha_i$, $\mathcal{F}$ is a simple direct summand of $Ind_{\alpha_1,\dots,\alpha_s}(\underline{\mathbb{C}})$

$$= p_*\left(\underline{\mathbb{C}}_{\widetilde{Coh}_{\alpha_1,\dots,\alpha_s}}\right).$$

$\to$ can replace $\mathcal{F}$ by $p_*(\underline{\mathbb{C}})$.

cotangent correspondence:

$$\begin{tikzcd} & T^*Coh_\alpha \times_{Coh_\alpha} \widetilde{Coh}_{\alpha_1,\dots,\alpha_s} \arrow[dl, "(dp)^*"'] \arrow[dr, "pr_1"] & \\ T^*\widetilde{Coh}_{\alpha_1,\dots,\alpha_s} & & T^*Coh_\alpha \end{tikzcd}$$

$$SS(\varphi_* \underline{\mathbb{C}}) \subset pr_1\left((d\varphi)^*\left(T^*_{\widetilde{Coh}}\widetilde{Coh}\right)\right)$$

$$= \left\{ (\mathcal{F}, \theta) \in Higgs_\alpha \;\middle|\; \begin{array}{l} \exists\, 0 = \mathcal{F}_s \subset \cdots \subset \mathcal{F}_0 = \mathcal{F} \\ [\mathcal{F}_{i-1} / \mathcal{F}_i] = \alpha_i, \\ \theta(\mathcal{F}_{i-1}) \subset \mathcal{F}_i \end{array} \right\}$$

$$\subset \Lambda_\alpha \qquad \blacksquare$$

Théorème (Sala - Schiffmann, Davison-H-Schiffmann) (Vasserot)

① CC is an algebra map

② CC is surjective

③ CC is bijective if $0 \leq g \leq 1$

Conjecture: • CC is always bijective

• CC is a bialgebra map.

$\Rightarrow$ spherical Eisenstein perverse sheaves are Verdier self dual.

$[\ CC(\mathbb{D}\mathcal{F}) = CC(\mathcal{F})\ ]$.

The conjecture is known in rank ≤ 1.

Proof of the theorem

① Algebra map: we use the local description of the product:

$$
\begin{tikzcd}
T^*X & Z^\circ \arrow[l, "\phi"'] \arrow[r, "\psi"] & T^*X' \\
 & T^*_{W^\circ}(X\times X') \arrow[u, phantom, "\shortparallel"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
T^*_{W^\circ}(X\times X') \arrow[r, "\pi_3"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\Gamma"] \arrow[rr, bend left, "\psi"] \arrow[dd, bend right=60, "\phi"'] & T^*X'\times_{X'} W^\circ \arrow[r, "\pi_4"] \arrow[d, "(dp)^*"] & T^*X' \\
T^*X\times_X W^\circ \arrow[r, "(dq)^*"] \arrow[d, "\pi_2"] & T^*W^\circ & \\
T^*X \arrow[uurr, bend right=40] & &
\end{tikzcd}
$$

$\mathcal{F}$ constructible sheaf on $X \Rightarrow CC(p_*q^*\mathcal{F}) =$ red path

The square is cartesian and has no excess intersection bundle

$\Rightarrow$ red path = green path

$$\Rightarrow CC(p_* q^* \mathcal{F}) = \psi_* \phi^! CC(\mathcal{F}) \qquad \blacksquare$$

② CC is surjective

* zero section of $Higgs_\alpha \longrightarrow Coh_\alpha = \pm CC(\underline{\mathbb{C}}_{Coh_\alpha})$ is in the image of CC
* they generate $CoHA_\Lambda^{top}(X)$

$\longrightarrow$ need the explicit parametrization of $Irr(\Lambda_\alpha)$ due to Bozec + induction.

③ CC is bijective if $0 \leq g \leq 1$

$g=0$: $\mathrm{Coh}_\alpha = \bigsqcup_{n_1+\dots+n_r=d} \dfrac{pt}{\mathrm{Aut}\left(\bigoplus_{i=1}^{r} \mathcal{O}(n_i)\right)}$. stratification

$\alpha=(r,d)$

$\quad \overset{\parallel}{\Xi(\underline{n})}$ $\quad \underline{n}=(n_1,\dots,n_r) \in \mathbb{Z}^r/\mathfrak{S}_r$.

. simple objects of $\mathcal{P}_\alpha$: $IC(\Xi(\underline{n}))$, $\underline{n}$

. irreducible components of Λ_α : $\overline{T^*_{\Xi(\underline{n})}\mathrm{Coh}_\alpha}$, $\underline{n}$.

$g=1$ ① Use the Harder-Narasimhan stratification of Coh_α for $\alpha \in \mathbb{Z}^+$: $\mathrm{Coh}_\alpha = \bigsqcup_{\substack{(\alpha_1,\dots,\alpha_s) \\ \mu(\alpha_1)>\dots>\mu(\alpha_s) \\ \sum \alpha_i = \alpha}} \mathrm{Coh}_{(\alpha_1,\dots,\alpha_s)}$

. Smooth morphism $\mathrm{Coh}_{\alpha_1,\dots,\alpha_s} \longrightarrow \prod_{i=1}^{s} \mathrm{Coh}(\alpha_i)$ thanks to unicity of the HN-filtration

- For an elliptic curve, $\mathrm{Coh}_{(\alpha)} \simeq \mathrm{Coh}_{(0,\gcd(\alpha))}$

- We know that cc is bijective in rank 0. ∎

In fact, in these cases, we can prove more: we have the unitriangularity of the cc map when we use the basis of simple perverse sheaves of $K_0(\mathcal{P})$ and the basis of irreducible components of Λ for $\mathrm{CoHA}^{top}_{\Lambda}(X)$.

Conjecture: the unitriangularity holds in general.

Thank you for your attention