Séminaire d'algèbre
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((Semi-) Canonical) bases of the elliptic Hall algebra

I - Canonical basis of quantum groups
- * Kac-Moody type quantum groups
- * perverse sheaves categorification

II - The elliptic Hall algebra (EHA): short reminder
- * elliptic Hall algebra
- * perverse sheaves categorification

III - The elliptic global nilpotent cone
- * irreducible components
- * a combinatorial problem.

I - Canonical basis of quantum groups

$Q = (\underbrace{I}_{\text{vertices}}, \underbrace{\Omega}_{\text{arrows}})$ (loop-free) quiver.

e.g. $Q = \bullet \rightrightarrows \bullet$

A Cartan matrix of Q (symmetric)

$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$\mathfrak{g}_Q = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ Kac-Moody algebra defined by generator and relations

$\cup$

$\mathfrak{n}_+$ positive part

$U(\mathfrak{n}_+)$ enveloping algebra: Chevalley generators E_i, $i \in I$ together with Serre relations

$\forall i \neq j \in I$, $$\sum_{k+\ell = 1-a_{ij}} \binom{1-a_{ij}}{k} E_i^k E_j E_i^\ell = 0$$

$U_q(\mathfrak{n}_+)$ positive part of the quantum group; $\mathbb{C}(q)$-algebra generators $E_i \in I$ and q-deformed Serre relations.

$$\binom{1-a_{ij}}{k}_q \quad \text{instead of} \quad \binom{1-a_{ij}}{k}.$$

$$\bigcup$$

$\boxed{U_q^{\mathbb{Z}}(\mathfrak{n}_+)}$ Lusztig integral form

$\mathbb{Z}[q^{\pm 1}]$-subalgebra of $U_q(\mathfrak{n}_+)$ generated by

$$\frac{E_i^\ell}{[\ell]_q!}, \quad i \in I, \ \ell \in \mathbb{N}.$$

Tremendous developments in the study of these quantum groups in the early 90's: Ringel, Lusztig, Kashiwara, ...

Lusztig categorification of $U_q^{\mathbb{Z}}(\mathfrak{n}_+)$ and the canonical basis

$Q=(I,\Omega)$ quiver

$\mathcal{M}_Q = \bigsqcup_{d\in\mathbb{N}^I} \mathcal{M}_{Q,d}$ (Artin) stack of representations

$d = d'+d''$

$$\begin{tikzcd} & \mathcal{E}xact_{d',d''} \arrow[dl, "q"', "\text{affine fibration}"] \arrow[dr, "p", "\text{projective morphism}"'] & \\ \mathcal{M}_{Q,d'}\times\mathcal{M}_{Q,d''} & & \mathcal{M}_{Q,d} \end{tikzcd}$$

smooth stack

$\mathcal{Q}$ = Lusztig category $\subset D_c^b(\mathcal{M})$ sub. additive category.

$= \bigoplus_{\ell\in\mathbb{Z}} \mathcal{P}[\ell]$

where $\mathcal{P}$ = semisimple category of perverse sheaves on $\mathcal{M}$ generated by the simple constituents of

$$p_* q^* \Big(p_* q^* \big(p_* q^* (\underline{\mathbb{C}}_{d_1} \boxtimes \mathbb{C}_{d_2}) \big) \boxtimes \mathbb{C}_{d_3} \ \cdots \Big) \boxtimes \underline{\mathbb{C}}_{d_s}$$

where d_j is concentrated at one vertex for $1\leq j\leq s$.

Convolution diagram $\Rightarrow$ associative multiplication on $K_\oplus(\mathcal{Q})$

shift $[1]$ $\Rightarrow$ $\mathbb{Z}[q^{\pm 1}]$-algebra structure on $K_\oplus(\mathcal{Q})$

Thm (Lusztig) $K_\oplus(\mathcal{Q}) \overset{\Phi}{\simeq} U_q^{\mathbb{Z}}(\mathfrak{n}_+)$ as $\mathbb{Z}[q^{\pm1}]$-algebras

$\cup$

$\mathcal{B} = \{ [\mathcal{F}], \mathcal{F}\in\mathcal{P} \text{ simple perverse sheaf} \}$ $\mathbb{Z}[q,q^{-1}]$-basis

$\mathcal{B}$ corresponds to Kashiwara global crystal basis under ϕ.

<u>Fact</u>: $K_0(\mathcal{P})$ is a cocommutative bialgebra

$$\begin{array}{c} K_0(\mathcal{P}) \\ \cong \\ U(\mathfrak{n}_+) \end{array}$$

Quantum groups associated to curves (Schiffmann)

<u>Key fact</u>: Rep Q, $Coh(C)$ are of homological dimension 1 if Q is a quiver

C is a (smooth projective) curve.

Combinatorial side: not well developped yet (recent work of Negut, combined with less recent work of Schiffmann-Vasserot)

Geometric side: work of Schiffmann.

$$\mathbb{Z}^+ = \{(r,d) \in \mathbb{Z}^2 \mid r > 0 \text{ or } r = 0, d \geq 0\}$$

(rank, degree) of coherent sheaves on a curve

$$Coh(C) = \bigsqcup_{\alpha \in \mathbb{Z}^+} Coh_\alpha(C) \quad \text{stack of coherent sheaves.}$$

Convolution diagram $\alpha', \alpha'' \in \mathbb{Z}^+$ $\alpha' + \alpha'' = \alpha$

$$\begin{tikzcd}
 & Exact_{\alpha',\alpha''}^{\text{smooth}} \arrow[dl, "q"', "\text{affine fibration}"] \arrow[dr, "p", "\text{projective}"'] & \\
Coh_{\alpha'}(C) \times Coh_{\alpha''}(C) & & Coh_\alpha(C)
\end{tikzcd}$$

Schiffmann defines

$$\mathcal{Q} = \bigoplus_{l \in \mathbb{Z}} \mathcal{P}[l] \subset D(\mathrm{Coh}\, C)$$

$\mathcal{P}$ = semisimple category of perverse sheaves on $\mathrm{Coh}(C)$ generated by the simple constituents of

= category of "spherical Eisenstein sheaves"

$$p_* q^* \left(\cdots p_* q^* \left(p_* q^* \left(\underline{\mathbb{C}}_{\alpha_1} \boxtimes \underline{\mathbb{C}}_{\alpha_2} \right) \boxtimes \underline{\mathbb{C}}_{\alpha_3} \right) \boxtimes \cdots \boxtimes \underline{\mathbb{C}}_{\alpha_s} \right)$$

for $\mathrm{rank}(\alpha_i) \leq 1$ that is $\alpha_i = (1, d_i)$ $d_i \in \mathbb{Z}$
or $(0, d_i)$ $d_i \geq 0$.

- $K_\oplus(\mathcal{Q})$ is a $\mathbb{Z}[q^{\pm 1}]$-algebra where q acts as $[1]$.

In fact, Schiffmann works over $\mathbb{F}_q$. In this case, it is possible to upgrade this to a $\mathbb{Z}[\sigma_1, \dots, \sigma_{2g}]$-algebra where the σ_i are indeterminates which are meant to represent the Weil numbers of C.

- We work over $\mathbb{C}$ so we loose this refined structure. We consider instead $K_0(\mathcal{Q})$.

Fact: • Using induction and restriction, $K_0(\mathcal{Q})$ is a (topological) bi-algebra

• It is expected to be co-commutative.

• It is known to be co-commutative if C is either a genus 0 or a genus 1 curve (or for weighted projective curves of genus $0 \leq g \leq 1$).

So $\mathcal{K}_0(\mathcal{Q}) = U(\mathfrak{n}_{\mathfrak{g}_C})$ for some Lie algebra $\mathfrak{g}_C$.
We want to understand bases of $\mathcal{K}_0(\mathcal{Q})$.
A first basis a given by $\{[\mathcal{F}] \mid \mathcal{F} \in \mathcal{P}$ simple perverse sheaf$\}$.
We get an other basis looking at the global nilpotent cone.

The (elliptic) global nilpotent cone

C smooth projective curve.

$$\mathcal{H}iggs = \bigsqcup_{\alpha \in \mathbb{Z}^+} \mathcal{H}iggs_\alpha \quad \text{stack of Higgs sheaves}$$

$$\left(\mathcal{F},\ \mathcal{F} \xrightarrow{\theta} \mathcal{F} \otimes \omega_C\right)$$

$\llcorner$ canonical bundle of the curve

$C = E$ elliptic curve $\Rightarrow$ $\omega_C = \mathcal{O}_C$ is the structure sheaf.

$$\mathcal{N} = \bigsqcup_{\alpha \in \mathbb{Z}^+} \mathcal{N}_\alpha \quad \text{global nilpotent cone}$$

$(\mathcal{F}, \theta)$ nilpotent if the composition

$$\mathcal{F} \xrightarrow{\theta} \mathcal{F} \otimes \omega_C \xrightarrow{\theta \otimes id} \mathcal{F} \otimes \omega_C^{\otimes 2} \to \cdots \to \mathcal{F} \otimes \omega_C^{\otimes s}$$

is 0 for $s \geq 0$.

Geometrico - combinatorial problem I

Describe the irreducible components of $\mathcal{N}_\alpha$, $\alpha \in \mathbb{Z}^+$.

General answer : Bozec, using Jordan types.

Jordan type of a nilpotent Higgs sheaf

$(\mathcal{F}, \theta)$ nilpotent Higgs sheaf.

$$\alpha_i = \left[\ker\left(\operatorname{im}\theta^{i-1}/\operatorname{im}\theta^{i}\otimes\omega_C^{-1} \to \operatorname{im}\theta^{i}/\operatorname{im}\theta^{i+1}\otimes\omega_C^{-1}\right)\right] \in \mathbb{Z}^+$$

$\underline{\alpha} = (\alpha_1, \dots, \alpha_s)$

$\mathcal{N}_{\underline{\alpha}}$ = Higgs sheaves of type $\underline{\alpha}$

$\overline{\mathcal{N}_{\underline{\alpha}}}$ is an irreducible component of $\mathcal{N}$.

$$JT(\alpha) = \left\{ \underline{\alpha} = (\alpha_1, \dots, \alpha_s) \;\middle|\; \sum_{k=1}^{s} \sum_{j \geq k} \alpha_j((k-j)\ell) = \alpha \right\}$$

indexes irreducible components of $\mathcal{N}_\alpha$.

Second parametrization for the elliptic nilpotent cone.

elliptic global nilpotent cone : global nilpotent cone of an elliptic curve E.

Combining

- Harder - Narasimhan stratification of $Coh(E)$
- $Coh^{ss}_{(r,d)}(E) \simeq Coh^{ss}_{(0, \mathrm{pgcd}(r,d))}(E)$

isomorphism of stacks

• Lusztig stratification of $Coh_{(0,d)}(E)$, $d \geq 0$,

we find strata $S \subset Coh_\alpha(E)$ parametrized by $\left((\alpha_1, \ldots, \alpha_s), \mu_1, \ldots, \mu_s : \mathcal{P} \to \mathbb{N}\right)$ (*)

HN-type

$\sum \alpha_i = \alpha$

$\mu(\alpha_1) > \ldots > \mu(\alpha_s)$

$$\sum \mu_i(\lambda)\,|\lambda| = \gcd(\alpha_i)$$

s.t. any irreducible component Λ of $\mathcal{N}$ can be written $\overline{T^*_S\, Coh_\alpha(E)}$ for some S.

The strata we need are the one for which $\mu_i(\lambda) = 0$ except if $\ell(\lambda) = 1$.

The only point specific to elliptic curves is the 2nd.

$\to$ goes back to Atiyah.

The stratification is a refinement of the Harder-Narasimhan filtration

If $\underline{\alpha} = (\alpha_1, \ldots, \alpha_s)$ is a Harder-Narasimhan type $|\underline{\alpha}| = \sum_{i=1}^{s} \alpha_i = \alpha$, we have a smooth map

$$\begin{array}{c} \mathrm{Coh}_{\underline{\alpha}}(E) \\ \Big\downarrow p \\ \prod_{i=1}^{s} \mathrm{Coh}^{ss}_{\alpha_i}(E) \end{array}$$

taking a coherent sheaf to the $\underbrace{\text{subquotients}}_{}$ (successive) subquotients of its HN filtration.

The refined strata are of the form

$$p^{-1}(S_1 \times \ldots \times S_s) \quad \text{for refined strata}$$

$$S_j \subset \mathrm{Coh}^{ss}_{\alpha_j}(E)$$

We only need to describe these.

→ We use the second point: it suffices to describe the refined strata of $\mathrm{Coh}^{ss}_{(0,d)}(E)$

→ We stratify $\mathrm{Coh}^{ss}_{(0,d)}(E)$ using Lusztig stratification.

Lusztig stratification = stratification of $\mathfrak{g}$ reductive Lie algebra

$\mathfrak{g} = \mathfrak{gl}_n$: Jordan stratification.

"Locally", $\mathrm{Coh}_{(0,d)} \simeq \mathfrak{gl}_d / GL_d$; the stratifications coincide under this isomorphism.

For $x \in C$, $\underset{\text{torsion sheaves supported at } x}{\mathrm{Tor}_x} \simeq \mathcal{O}_x$ - modules, $\mathcal{O}_x$ is a discrete valuation ring
$\Downarrow$
$\mathcal{O}_x$- modules are classified by partitions $\longrightarrow M_{x,\lambda}$, $\lambda \in \mathcal{P}$

More explicitly, for $\mu : \mathcal{P} \longrightarrow \mathbb{N}$, take $\lambda_1, \dots, \lambda_t \in \mathcal{P}$ s.t. in this list, $\lambda \in \mathcal{P}$ appears $\mu(\lambda)$-times.

S_μ consists of torsion sheaves isomorphic to a direct sum $\bigoplus_{i=1}^{t} M_{x_i, \lambda_i}$ for pairwise distinct $x_i \in C$.

Consider the projection
$$\begin{array}{c} \mathcal{N}_\alpha \\ \pi_\alpha \downarrow \\ Coh_\alpha(E) \end{array}.$$

- $\dim S$, $S \in \mathcal{S}$ is known
- π_α is smooth over each S and the fiber dimension is known [using arguments of Ringel]

$\Rightarrow \dim \pi^{-1}(S)$ is known

- $\mathcal{N}_\alpha$ is Lagrangian inside $Higgs_\alpha$

$\rightarrow$ description of irreducible components of $\mathcal{N}_\alpha$.

Geometrico-combinatorial problem II

- Relate the two parametrization.
- (valid for any curve)

  ① find the generic HN-type of the irreducible component $\mathcal{N}_{\underline{\alpha}} \subset \mathcal{N}$

  ② find the generic HN-type of the projection $\pi_\alpha(\mathcal{N}_{\underline{\alpha}})$.

$\longrightarrow$ calculations in small dimensions let piecewise linear structures appear. (*)

Back to $\mathcal{U}(\mathfrak{g}_c)$

Link between $\mathcal{P}$ perverse sheaves and
$\mathcal{N}$ global nilpotent cone

$\rightarrow$ Characteristic cycle map.

completions

$$CC : \widehat{K_0(\mathcal{P})} \longrightarrow \widehat{\mathbb{Z}[\mathrm{Irr}\,\mathcal{N}]}$$

mysterious map, rather hard to describe.

<u>But</u>: If C is an elliptic curve, it is an isomorphism (of algebras when the r.h.s. is endowed with the (restriction) of the CoHA product)

Under CC, the two bases are <u>not</u> the same.

Question: describe the base change matrix.

that is describe the microlocal multiplicities of the simple perverse sheaves $\mathcal{F} \in \mathcal{P}$.

- On an open of the support $\rightarrow$ product of Kostka numbers.

Not computed yet on the whole stack of coherent sheaves.

related to the value at 1 of Schiffmann's elliptic Kostka numbers? (defined in terms of change of basis matrix of 2 bases of the elliptic Hall algebra.

Example of correspondence Jordan types — HN type

$C = E$ elliptic curve

Jordan type		HN-type
(α)		(α)
$((0,0),(0,0),(1,e))$		(α) $\quad \alpha=(3,3e)$
$((1,e),(1,e))$		(α), $\quad \alpha=(3,3e)$
$((1,e_1),(1,e_2))$	$e_2 > e_1$	$((1,e_2),(2,e_1+e_2))$
$((1,e_1),(1,e_2))$	$e_1 > e_2$	$((2,e_1+e_2),(1,e_2))$
$((0,e_1),(0,0),(1,e_2))$	$e_1 > 0$	$((1,e_1+e_2),(2,2e_2))$
$((0,0),(0,e_1),(1,e_2))$	$e_1 > 0$	$((2,2(e_1+e_2)),(1,e_2))$
$((0,e_1),(0,e_2),(1,e_3))$	$e_1 > 0$, $e_2 > 0$	$((1,e_1+e_2+e_3),(1,e_2+e_3),(1,e_3))$