Student projects

Symmetric and general linear groups in complex rank: Deligne categories

Project ref: UG1175

Project supervisor: Lucien Hennecart

Meets requirements for

Project description

Key-words: symmetric group, linear group, representation theory, Abelian categories, tensor categories, Karoubian categories

The symmetric group S_n (permutations of a set with n elements) and the general linear group GL_n (symmetries of an n-dimensional vector space) are ubiquitous objects in representation theory. The first is the Weyl group of the second, which gives deep connections between their respective representation theories.

Deligne and Milne introduced in the 1980s a 1-parameter deformation of their categories of representations, Rep(S_t) and Rep(GL_t), where t is a (non-necessary integer) complex number. This led to the subfield of representation theory known as "representation theory in complex rank", and a program to develop the subject was proposed by Etingof around 2014. It appears that any theorems that hold for nonnegative integer rank can be adapted to the case of complex ranks (description of indecomposable objects, dimension formulas), but new phenomena appear. The definition of these new categories is rather elementary, but they enjoy deep and interesting behaviours. It is possible to draw diagrams (graphical calculus) to visualise the objects under consideration.

One of the motivations of Deligne and Milne was to find "exotic" symmetric tensor categories, i.e. symmetric tensor categories which are not representations of a (super)group.

The goal of this project is (1) to understand the representation theory of the symmetric groups and the general linear groups (2) to study the corresponding categories of representations, in particular, their tensor structure (3) to define Deligne categories using categorical tools and study their properties.

There are other topics worth considering, such as the interpolation of other families of categories, Schur-Weyl duality, Tannakian formalism, and many more.

Some references include:

[1] Deligne, Pierre. "La catégorie des représentations du groupe symétrique St, lorsque t n'est pas un entier naturel." *Algebraic groups and homogeneous spaces* 19 (2007): 209-273.

[2] Entova Aizenbud, Inna. "Schur?Weyl duality for Deligne categories." *International Mathematics Research Notices* 2015.18 (2015): 8959-9060.

[3] Etingof, Pavel, et al. *Tensor categories*. Vol. 205. American Mathematical Soc., 2016.

[4] Etingof, Pavel. "Representation theory in complex rank, I." *Transformation Groups* 19 (2014): 359-381.

[5] Etingof, Pavel, and Arun S. Kannan. "Lectures on symmetric tensor categories." *arXiv preprint arXiv:2103.04878* (2021).

[6] Harman, Nate, and Daniil Kalinov. "Classification of simple algebras in the Deligne category Rep (St)." *Journal of Algebra* 549 (2020): 215-248.

[7] Knop, Friedrich. "A construction of semisimple tensor categories." *Comptes Rendus Mathematique* 343.1 (2006): 15-18.

Prerequisites

Being interested in (learning) algebra, representation theory, and category theory.

Recommended reading

To get an idea of the kind of math involved (certainly challenging in the first reading): Etingof, Pavel, and Arun S. Kannan. "Lectures on symmetric tensor categories." arXiv preprint arXiv:2103.04878 (2021).

Very nice lectures on representation theory by Etingof: https://math.mit.edu/~etingof/repb.pdf.
A fairly well-determined amount of them will be useful for us, in particular the Schur-Weyl duality between representations of symmetric and general linear groups. I would recommend first reading Chapters 1 and 2.