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Nonabelian Hodge correspondence and cohomological Hall algebras

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Nonabelian Hodge theory (NAHT)

Nonabelian Hodge theory is a deep relationship between 3 kinds of objects on a smooth projective variety X:

1. Local systems
$$\pi_1(X,x) \to \mathrm{GL}_r(\mathbf{C})$$
2. (semistable) Higgs bundles,
$$(\mathcal{F},\theta),\ \text{with } \theta\colon \mathcal{F} \to \mathcal{F}\otimes\Omega^1_X\ \mathcal{O}_X\text{-linear and } (\theta\otimes \mathrm{id}_{\Omega^1_X})\circ\theta = 0$$
   $+$ some vanishing condition for Chern classes (integrability automatic for curves, and degree 0).
3. flat connections
$$(\mathcal{F},\theta),\ \text{with } \theta\colon \mathcal{F} \to \mathcal{F}\otimes\Omega^1_X\ \text{Leibniz rule and } (\theta\otimes \mathrm{id}_{\Omega^1_X})\circ\theta = 0$$

In this talk, we concentrate on X smooth projective curve. This is already a highly interesting and nontrivial situation, framework of the celebrated $P = W$ conjecture (now proven twice).

NAHT – Categories

1. For local systems, $\mathrm{Rep}(\pi_1(C,x))$, where $x \in C$ is some fixed point,
2. For Higgs bundles, the category of semistable Higgs bundles of degree 0 (with some vanishing condition for Chern classes and integrability),
   semistability: for any $\mathcal{G} \subset \mathcal{F}$ such that $\theta(\mathcal{G}) \subset \mathcal{G} \otimes \Omega^1_C$, $\deg(\mathcal{F}) \leq 0$.
3. For connections, the category of vector bundles with flat connection.

Simpson (1990s): All three categories are equivalent, and equivalent to some category of *harmonic bundles* (do not appear again in this talk).

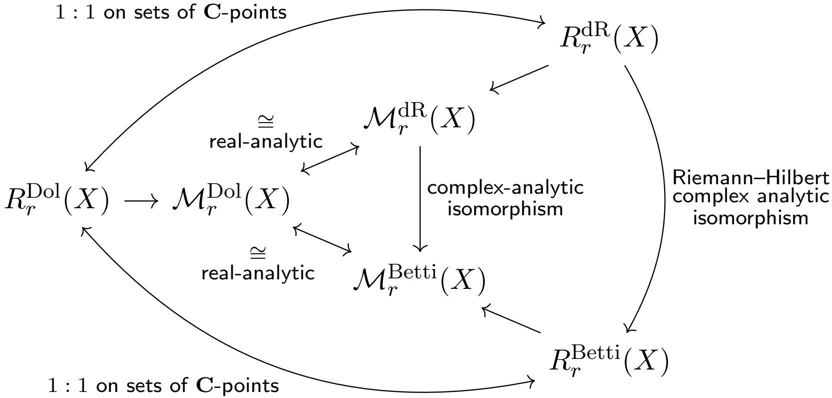
NAHT – Moduli spaces

Constructing moduli spaces is a geometric problem. One proceeds as follows:

1. Consider framed objects
2. Construct the moduli space of framed objects
3. Take the quotient by the algebraic group changing the framing.

Framed objects have a fine moduli space: we can construct $R_r^\sharp(C)$ for $\sharp \in \{\mathrm{Dol}, \mathrm{Betti}, \mathrm{dR}\}$.
By taking the GIT quotient by the GL_r-action, we obtain the moduli spaces $\mathcal{M}_r^\sharp(C)$.
Considering instead the stacky quotients, we obtain the moduli stacks $\mathfrak{M}_r^\sharp(C)$.
Framed connections/Higgs bundles/local systems: one chooses a point $x \in X$ and pairs $(\mathcal{F}, \beta\colon \mathcal{F}_x \to V)$ of an object together with a isomorphism of the stalk at x with a fixed vector space.

NATH – Homeomorphisms and Simpson's questions



Question (Simpson): Can we compare the GL_r-equivariant geometry of the spaces $R_r^{\sharp}$?

Today: How to answer the question positively.

Illustration

Let C be a smooth projective curve of genus $g \geq 1$.

1. $\mathcal{M}_r^{\mathrm{Betti}} \cong (\mathbf{C}^\times)^{2g} \cong (S^1)^{2g} \times (\mathbf{R}_+^\times)^{2g}$
2. $\mathcal{M}_r^{\mathrm{Dol}} \cong \mathrm{T}^*\mathrm{Jac}(C)$

These algebraic varieties are indeed homeomorphic, but not by an algebraic morphism:

1. $\mathcal{M}_r^{\mathrm{Betti}}$ is an affine algebraic variety,
2. $\mathcal{M}_r^{\mathrm{Dol}}$ receives a nonconstant from a smooth projective algebraic variety, $\mathrm{Jac}(C)$, and so cannot be affine.

NAHT – Stacks

By construction, only global quotient stacks appear! We are only studying the equivariant geometry of some spaces.
If $\mathfrak{M} = X/G$ is the stacky quotient of an algebraic variety X acted on by a group G,

$$\mathrm{H}^*(\mathfrak{M}) = \mathrm{H}^*_G(X), \quad \mathrm{H}^{\mathrm{BM}}_*(\mathfrak{M}) = \mathrm{H}^{\mathrm{BM},G}_*(X).$$

Computing these (co)homology groups is done via the Borel construction: we let E be a contractible G space. Then, we set for example

$$\mathrm{H}^*_G(X) = \mathrm{H}^*(X \times_G E).$$

For example, if $G = \mathbf{C}^*$, one can take $E = \mathbf{C}^\infty \setminus \{0\}$.

Cohomological Hall algebras

Cohomological Hall algebras are an extremely powerful tool to study the Borel–Moore homology of moduli stacks of objects in categories.

We construct an algebra structure on the spaces $\mathrm{H}_*^{\mathrm{BM}}(\mathfrak{M}^\sharp)$.

We enrich this vector space with more algebraic structure to make it easier to study.

The key diagram is

$$\mathfrak{M}^\sharp \times \mathfrak{M}^\sharp \xleftarrow{q^\sharp} \mathfrak{Exact}^\sharp \xrightarrow{p^\sharp} \mathfrak{M}^\sharp$$

The key properties are the quasi-smoothness of $q^\sharp$ and the properness of $p^\sharp$.

Pull-push along this diagram provides us with the CoHA product

$$m \colon \mathrm{H}_*^{\mathrm{BM}}(\mathfrak{M}^\sharp) \otimes \mathrm{H}_*^{\mathrm{BM}}(\mathfrak{M}^\sharp) \to \mathrm{H}_*^{\mathrm{BM}}(\mathfrak{M}^\sharp).$$

Virtual pullback

The morphism $q^\sharp\colon \mathfrak{Exact}^\sharp \to \mathfrak{M}^\sharp \times \mathfrak{M}^\sharp$ has a very favourable structure.
In technical terms: it is the classical truncation of the total space of its tangent complex.
Explicitly: there is a 3-term complex of vector bundles on $\mathfrak{M}^\sharp \times \mathfrak{M}^\sharp$:

$$V^{-1} \xrightarrow{d^{-1}} V^0 \xrightarrow{d^0} V^1.$$

It induces a *vector bundle stack* V^0/V^{-1} on $\mathfrak{M}^\sharp \times \mathfrak{M}^\sharp$, a vector bundles $\tilde{\pi}_0^* V^1$ on V^0/V^{-1} and d^0 gives a section

$$s_{d^0}\colon V^0/V^{-1} \to \tilde{\pi}_0^* V^1.$$

Fact: $\mathfrak{Exact}^\sharp$ is canonically identified with $\{s_{d^0} = 0\}$.
Virtual pull-back is now done in two steps.

CoHAs and BPS algebra

The Borel–Moore homology $\mathrm{H}^{\mathrm{BM}}_*(\mathfrak{M}^\sharp)$ is the cohomology of the complex of mixed Hodge modules $\mathtt{JH}_*\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{\mathfrak{M}^\sharp}$.

Proposition (Davison–H–Schlegel Mejia)

$\mathtt{JH}_*\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{\mathfrak{M}^\sharp}$ *is concentrated in nonnegative cohomological degrees.*

The *relative BPS algebra* is defined as the degree 0 cohomology:

$$\mathcal{BPS}^\sharp_{\mathrm{Alg}} := \mathcal{H}^0(\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{\mathfrak{M}^\sharp}).$$

This is an algebra object in $\mathrm{MHM}(\mathcal{M}^\sharp)$.
The monoidal structure on $\mathcal{D}^+(\mathrm{MHM}(\mathcal{M}^\sharp))$ is given by

$$\mathscr{F} \boxdot \mathscr{G} := \oplus_*(\mathscr{F} \boxtimes \mathscr{G}).$$

where $\oplus\colon \mathcal{M}^\sharp \times \mathcal{M}^\sharp \to \mathcal{M}^\sharp$ is the direct sum.

CoHAs and generalised Kac–Moody Lie algebras

Let C be a smooth projective curve of genus ≥ 2 (genus 0, 1 are treated separatly; the picture is slightly different but easy).

Theorem (Davison–H–Schlegel Mejia, H)

We have for $\sharp \in \{\mathrm{Dol}, \mathrm{Betti}, \mathrm{dR}\}$,

1. $\mathcal{BPS}^{\sharp}_{\mathrm{Alg}} \cong \mathrm{Free}_{\boxdot-\mathrm{Alg}}\left(\bigoplus_{r\geq 1}\mathcal{IC}(\mathcal{M}^{\sharp}_r)\right)$ *(isomorphism of algebra objects).*
2. $\mathrm{JH}_*\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{\mathfrak{M}^{\sharp}} = \mathrm{Sym}_{\boxdot-\mathrm{Alg}}\left(\mathcal{BPS}^{\sharp}_{\mathrm{Lie}} \otimes \mathrm{H}^*_{\mathbf{C}^*}\right)$ *(isomorphism of complexes of mixed Hodge modules) where* $\mathcal{BPS}^{\sharp}_{\mathrm{Lie}} := \mathrm{Free}_{\boxdot-\mathrm{Lie}}(\bigoplus_{r\geq 1}\mathcal{IC}(\mathcal{M}^{\sharp}_r))$.

We see that in the categories of constructible complexes, the Betti/Dolbeault/de Rham BPS algebras coincide as algebra objects and the full CoHAs coincide. More work is needed to compare the algebra structures for the full CoHAs.

Comparison of Betti and de Rham

The comparison of the Betti and de Rham CoHA is rather straightforward using the *derived Riemann–Hilbert correspondence* (Porta, Porta–Sala). It gives an isomorphism of CoHA diagrams between the *derived* moduli stacks. In particular, we have an isomorphism of the relative cotangent complexes $\mathbf{\Phi}^*\mathbb{L}_{q^{\mathrm{Betti}}} \cong \mathbb{L}_{q^{\mathrm{dR}}}$. Fom our construction, the virtual pullbacks therefore coincide and so do the CoHA products.

Theorem

The cohomological Hall algebras $\mathrm{H}_*^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Betti}})$ *and* $\mathrm{H}_*^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{dR}})$ *are canonically isomorphic.*

Comparison of de Rham and Dolbeault – Hodge–Deligne moduli space

To give an interpolation between the de Rham and Dolbeault moduli spaces, Deligne suggested to consider λ-connections. A λ-connection is (E, ∇) where $\nabla\colon E \to E \otimes \Omega^1_C$ satisfies the twisted Leibniz identity:

$$\nabla(f \cdot e) = \lambda(e \otimes \mathrm{d}f) + f\nabla(e).$$

We can construct the moduli space of framed rank r λ-connections R_r^{Hodge}, consider the quotient stack $\mathfrak{M}_r^{\mathrm{Hodge}} := R_r^{\mathrm{Hodge}}/\mathrm{GL}_r$ and the GIT quotient $\mathcal{M}_r^{\mathrm{Hodge}} := R_r^{\mathrm{Hodge}} /\!\!/ \mathrm{GL}_r$. The parameter λ gives morphisms

$$\mathfrak{M}^{\mathrm{Hod}} \to \mathcal{M}^{\mathrm{Hod}} \to \mathbf{A}^1.$$

By framed, we mean as usual that we add the datum of an isomorphism of the fiber at a fixed point with a vector space.

Comparison of de Rham and Dolbeault

Theorem

The constructible complex $p_*^{\mathrm{Hodge}}\mathbb{D}\mathbf{Q}_{\mathfrak{M}^{\mathrm{Hodge}}}$ *has locally constant cohomology sheaves where* $p^{\mathrm{Hodge}}\colon \mathfrak{M}^{\mathrm{Hodge}} \to \mathbf{A}^1$*.*

Since $\mathbf{A}^1$ is contractible, for any point $x \in \mathbf{A}^1$, $p_*^{\mathrm{Hodge}}\mathbb{D}\mathbf{Q}_{\mathfrak{M}^{\mathrm{Hodge}}} \cong \pi^*(p_*^{\mathrm{Hodge}}\mathbb{D}\mathbf{Q}_{\mathfrak{M}^{\mathrm{Hodge}}})_x$ for $\pi\colon \mathbf{A}^1 \to \mathrm{pt}$ and this is an identification of algebra objects since $\pi^*\colon \mathcal{D}_c^+(\mathrm{pt}) \to \mathcal{D}_c^+(\mathbf{A}^1)$ is an equivalence of categories.

Corollary

The cohomological Hall algebras $\mathrm{H}_*^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{dR}})$ *and* $\mathrm{H}_*^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}})$ *are canonically isomorphic.*

In other words, for curves, we have a cohomological Hall algebras enhancement of the nonabelian Hodge isomorphisms between cohomologies of the Dolbeault, Betti and de Rham moduli spaces.

CoHA for the Hodge–Deligne moduli space – Ideas of proof

Recall that we have the Hodge moduli stack, moduli space

$$\mathfrak{M}^{\mathrm{Hod}} \xrightarrow{\mathtt{JH}^{\mathrm{Hod}}} \mathcal{M}^{\mathrm{Hod}} \xrightarrow{p} \mathbf{A}^1.$$

As for the Betti, de Rham and Dolbeault situation, one can construct a relative CoHA product on

$$\mathscr{A}^{\mathrm{Hod}} := \mathtt{JH}^{\mathrm{Hod}}_{*}\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{\mathfrak{M}^{\mathrm{Hod}}}$$

1. Restricting over $\lambda \in \mathbf{A}^1$, we recover the cohomological for the stack of λ-connection.
2. $\mathscr{A}^{\mathrm{Hod}}$ is concentrated in nonnegative *relative* perverse degrees
3. $\mathscr{A}^{\mathrm{Hod}}$ is concentrated in *absolute* perverse degrees ≥ -2
4. As a complex of mixed Hodge modules, $\mathscr{A}^{\mathrm{Hod}}$ has nonnegative weights.

CoHA for the Hodge–Deligne moduli space continued

1. For any $r \geq 1$, and any $x \in \mathbf{A}^1$, we have $i_x^! \mathcal{IC}(\mathcal{M}^{\mathrm{Hod}})[2] = \mathcal{IC}(\mathcal{M}_x^{\mathrm{Hod}})$, where $\imath_x \colon \mathcal{M}_x^{\mathrm{Hod}} \to \mathcal{M}^{\mathrm{Hod}}$
2. For any $r \geq 1$, there is a canonical morphism

$$\mathcal{IC}(\mathcal{M}^{\mathrm{Hod}}/\mathbf{A}^1) := \mathcal{IC}(\mathcal{M}_r^{\mathrm{Hod}})[2] \to \mathscr{A}^{\mathrm{Hod}}$$

We can therefore construct a morphism

$$\mathrm{Free}_{\boxdot_{\mathbf{A}^1}}\left(\bigoplus_{r\geq 1} \mathcal{IC}(\mathcal{M}_r^{\mathrm{Hod}}/\mathbf{A}^1) \otimes \mathrm{H}^*_{\mathbf{C}^*}\right) \to \mathscr{A}^{\mathrm{Hod}}.$$

Theorem

This is an isomorphism.

Proof. We let $\mathcal{C}$ be the cone of this morphism. We have to prove it vanishes. It suffices to prove that it vanishes at any point. This follows from the fact that the restriction of this morphism for any $\lambda \in \mathbf{A}^1$ is an isomorphism by [DHS].

Further developments

1. Objects with parabolic structure
2. Nonabelian Hodge correspondence in positive characteristics and cohomological Hall algebras
3. Several other applications of the relative cohomological Hall algebras and relative perverse t-structures:
    1. Schiffmann positivity conjecture for counting polynomials of absolutely indecomposable vector bundles over a smooth projective curve
    2. deformation invariance of the BPS cohomology for quiver.