BPS algebras and generalised Kac–Moody algebras from 2-Calabi–Yau categories

Lucien Hennecart (lucien.hennecart@gmail.com)
joint work with Ben Davison and Sebastian Schlegel Mejia

1. Introduction

2-Calabi–Yau (CY) categories feature prominently in algebraic geometry and representation theory:

1. **semistable sheaves** on K3 or Abelian surfaces,
2. **semistable Higgs sheaves** on smooth projective curves,
3. representations of **preprojective algebras** of quivers,
4. representations of **fundamental groups of Riemann surfaces**.

We are interested in the **topology and singularities of the moduli stacks** and the good moduli spaces of objects in these categories. Our aim is to understand the **Borel–Moore homologies** of these geometric objects. We achieve this goal in three steps.

1. we define a **sheaf-theoretic cohomological Hall algebra** for a class of Abelian categories of dimension at most two,
2. we define the **BPS Lie algebra**, by generators and relations,
3. we relate the BPS Lie algebra to the whole cohomological Hall algebra through a **PBW theorem**.

We obtain

1. the **cohomological integrality** of all categories involved,
2. a **stacky nonabelian Hodge isomorphism** for curves,
3. the **positivity of cuspidal polynomials** of quivers (a strengthening of Kac positivity conjecture),
4. a lowest weight vector description for the **cohomology of Nakajima quiver varieties**.

2. 2-dimensional categories

The major examples of 2-CY categories we will be interested in are:

1. **Preprojective algebras of quivers**

   $Q=(Q_0,Q_1)$ quiver, $\overline{Q}=(Q_0,Q_1\sqcup Q_1^*)$ its double,
   $\rho=\sum_{\alpha\in Q_1}[\alpha,\alpha^*]$ the preprojective relation,
   $\Pi_Q=\mathbf{C}\overline{Q}/\rho$ the preprojective algebra.
2. **Semistable sheaves on K3 and Abelian surfaces**

   S symplectic surface, H polarisation, $\boldsymbol{v}\in \mathrm{H}^{\mathrm{even}}(S,\mathbf{Z})$ primitive Mukai vector
   $\mathrm{Coh}_{\boldsymbol{v}}^{H-\mathrm{ss}}(S)$ category of H-semistable sheaves on S with Mukai vector in $\mathbf{N}\boldsymbol{v}$.
3. **Semistable Higgs sheaves on smooth projective curves**

   C smooth projective curve, $\mu\in\mathbf{Q}$ slope
   $\theta\colon\mathcal{F}\to\mathcal{F}\otimes_{\mathcal{O}_C}K_C$ Higgs sheaf
   $\mathrm{Higgs}^{\mu-\mathrm{ss}}(C)$ category of semistable Higgs sheaves of slope μ.
4. (Twisted) **fundamental group algebras of Riemann surfaces**

   S (closed) Riemann surface, ξ root of unity
   $G=\langle\lambda,x_i,y_i:1\leq i\leq g\mid \lambda\prod x_iy_ix_i^{-1}y_i^{-1}=1\rangle=\pi_1(S\setminus\{\mathrm{pt}\})$
   $A=\mathbf{C}G/\langle\xi-\lambda\rangle$ twisted fundamental group algebra

3. Cohomological Hall algebras for 2-dimensional categories

Let $\mathcal{A}$ be a d-dimensional Abelian category ($d\leq 2$) and $\mathtt{JH}\colon\mathfrak{M}_{\mathcal{A}}\to\mathcal{M}_{\mathcal{A}}$ the Jordan–Hölder (semisimplification) map from the stack of objects to the good moduli space. The formula $\mathscr{F}\boxdot\mathscr{G}:=\oplus_*(\mathscr{F}\boxtimes\mathscr{G})$ gives $\mathcal{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathcal{A}}))$ a monoidal product, where we denote by $\oplus\colon\mathcal{M}_{\mathcal{A}}\times\mathcal{M}_{\mathcal{A}}\to\mathcal{M}_{\mathcal{A}}$ the direct sum.

Theorem. *The complex of mixed Hodge modules* $\mathscr{A}_{\mathcal{A}}:=\mathtt{JH}_*\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathcal{A}}}\in\mathcal{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathcal{A}}))$ *admits a (relative) cohomological Hall algebra structure.*

This algebra structure is constructed using the commutative diagram

$$\begin{array}{ccccc}\mathfrak{M}_{\mathcal{A}}\times\mathfrak{M}_{\mathcal{A}} & \xleftarrow{q} & \mathfrak{Exact}_{\mathcal{A}} & \xrightarrow{p} & \mathfrak{M}_{\mathcal{A}}\\ \downarrow{\scriptstyle \mathtt{JH}\times\mathtt{JH}} & & & & \downarrow{\scriptstyle \mathtt{JH}}\\ \mathcal{M}_{\mathcal{A}}\times\mathcal{M}_{\mathcal{A}} & & \xrightarrow[\oplus]{\qquad\qquad} & & \mathcal{M}_{\mathcal{A}}\end{array}$$

Key-facts. *1. The RHom complex over $\mathfrak{M}_{\mathcal{A}}\times\mathfrak{M}_{\mathcal{A}}$ can be represented by a 3-term complex of vector bundles. With $\mathcal{C}=\mathrm{RHom}[1]$, the map $q\colon\mathfrak{Exact}_{\mathcal{A}}=\mathrm{Tot}(\mathcal{C})\to\mathfrak{M}_{\mathcal{A}}\times\mathfrak{M}_{\mathcal{A}}$ gives a canonical virtual pullback map $\mathbb{D}\mathbf{Q}_{\mathfrak{M}_{\mathcal{A}}\times\mathfrak{M}_{\mathcal{A}}}\to q_*(\mathbb{D}\mathbf{Q}_{\mathfrak{Exact}_{\mathcal{A}}})[2(-,-)_{\mathcal{A}}]$.*

2. The map p is proper.

4. The BPS Lie algebra

Roots. The monoid of connected components of $\mathcal{M}_{\mathcal{A}}$ has the bilinear form induced by the Euler form $(-,-)$. We have the set of primitive positive roots

$$\Sigma_{\mathcal{A}}:=\{a\in\pi_0(\mathcal{M}_{\mathcal{A}})\mid\mathcal{M}_{\mathcal{A},a}\text{ contains simples}\}$$

and the set of positive roots $\Phi_{\mathcal{A}}^+:=\Sigma_{\mathcal{A}}\cup\{la\mid l\in\mathbf{N},a\in\Sigma_{\mathcal{A}},(a,a)=0\}$.

Generators. For $a\in\Sigma_{\mathcal{A}}$, we let $\mathscr{G}_{\mathcal{A},a}:=\mathcal{IC}(\mathcal{M}_{\mathcal{A},a})$. For $a\in\Sigma_{\mathcal{A}}$, $(a,a)=0$ and $l\geq 2$, we let $\mathscr{G}_{\mathcal{A},a}:=(u_m)_*\mathcal{IC}(\mathcal{M}_{\mathcal{A},a})$ where $u_m\colon\mathcal{M}_{\mathcal{A},a}\to\mathcal{M}_{\mathcal{A},la}, x\mapsto x^{\oplus l}$.

The BPS Lie algebra. The (relative) BPS Lie algebra is the Lie algebra object $\mathcal{BPS}_{\mathcal{A},\mathrm{Lie}}=\mathfrak{n}^+_{\pi_0(\mathcal{M}_{\mathcal{A}}),\mathscr{G}_{\mathcal{A}}}\in\mathrm{MHM}(\mathcal{M}_{\mathcal{A}})$ generated by $\mathscr{G}_{\mathcal{A},a}$, $a\in\Phi^+_{\mathcal{A}}$, modulo the relations

1. $\mathrm{ad}(\mathscr{G}_{\mathcal{A},a})(\mathscr{G}_{\mathcal{A},b})=0$ if $(a,b)=0$,
2. $\mathrm{ad}(\mathscr{G}_{\mathcal{A},a})^{1-(a,b)}(\mathscr{G}_{\mathcal{A},b})=0$ if $(a,a)=2$.

The BPS algebra. The (relative) BPS algebra is defined as $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}:=\mathcal{H}^0(\mathscr{A}_{\mathcal{A}})\in\mathrm{MHM}(\mathcal{M}_{\mathcal{A}})$.

Theorem. *We have a canonical isomorphism of algebras* $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}\cong\mathbf{U}(\mathfrak{n}^+_{\pi_0(\mathcal{M}_{\mathcal{A}}),\mathscr{G}_{\mathcal{A}}})$.

5. The PBW isomorphism

Theorem. *We have a **PBW isomorphism** in $\mathcal{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathcal{A}}))$:*

$$\mathrm{Sym}_{\boxdot}(\mathcal{BPS}_{\mathcal{A},\mathrm{Lie}}\otimes\mathrm{H}^*(\mathrm{B}\mathbf{C}^*))\to\mathscr{A}_{\mathcal{A}}.$$

In particular, we have **cohomological integrality** for the category $\mathcal{A}$.

6. Nonabelian Hodge isomorphism for stacks

Let C be a genus g smooth projective curve and $(r,d)\in\mathbf{Z}_{\geq 1}\times\mathbf{Z}$. Classical NAHT provides us with a diagram in which the middle arrow is an homeomorphism:

$$\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C)\xrightarrow{\mathtt{JH}}\mathcal{M}^{\mathrm{Dol}}_{r,d}\xrightarrow{\Psi}\mathcal{M}^{\mathrm{Betti}}_{g,r,d}\xleftarrow{\mathtt{JH}}\mathfrak{M}^{\mathrm{Betti}}_{g,r,d}.$$

Theorem. *We have a canonical isomorphism of constructible complexes*

$$\Psi_*\mathtt{JH}_*\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C)}\cong\mathtt{JH}_*\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{\mathfrak{M}^{\mathrm{Betti}}_{g,r,d}}$$

and, in particular, a canonical isomorphism in Borel–Moore homology:

$$\mathrm{H}^{\mathrm{BM}}_*(\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C))\cong\mathrm{H}^{\mathrm{BM}}_*(\mathfrak{M}^{\mathrm{Betti}}_{g,r,d})$$

7. Positivity of cuspidal polynomials

Let Q be a quiver, $\mathbf{F}_q$ a finite field and $H_{Q,\mathbf{F}_q}$ the Hall algebra of Q over $\mathbf{F}_q$. This is a $\mathbf{N}^{Q_0}$-graded twisted bialgebra. Its primitive elements are called *cuspidal functions*:

$$H^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\boldsymbol{d}]:=\{f\in H_{Q,\mathbf{F}_q}\mid\Delta(f)=f\otimes 1+1\otimes f\},$$

and $C_{Q,\boldsymbol{d}}(q)$ denotes its dimension.

Theorem. *For $\mathbf{d}\in\Sigma_{\Pi_Q}$, we have $C_{Q,\mathbf{d}}(q^{-2})=\mathrm{IP}(\mathcal{M}_{\Pi_Q,\mathbf{d}})$, and so $C_{Q,\mathbf{d}}(q)\in\mathbf{N}[q]$.*

This gives a way to compute the intersection cohomology of all Nakajima quiver varieties, using the Borcherds–Kac–Weyl character formula for generalised Kac–Moody algebras.

8. Decomposition of the cohomology of Nakajima quiver varieties

Let $N_{Q,\mathbf{f},\mathbf{d}}$ be the Nakajima quiver variety for the quiver Q and framing data $\mathbf{f}$. We let $\mathbb{M}_{\mathbf{f}}(Q):=\bigoplus_{\mathbf{d}\in\mathbf{N}^{Q_0}}\mathrm{H}^*(N_{Q,\mathbf{f},\mathbf{d}},\mathbf{Q}^{\mathrm{vir}})$. This is a representation of the Lie algebra $\mathfrak{g}^{\mathrm{BPS}}_{\Pi_Q}$ (the double of $\mathfrak{n}^{\mathrm{BPS},+}_{\Pi_Q}:=\mathrm{H}^*(\mathcal{BPS}_{\Pi_Q,\mathrm{Lie}})$).

Theorem. *We have the decomposition*

$$\mathbb{M}_{\mathbf{f}}(Q)=\bigoplus\nolimits_{(\mathbf{d},1)\in\Sigma_{\Pi_{Q_{\mathbf{f}}}}}\mathrm{IH}^*(N(Q,\mathbf{f},\mathbf{d}))\otimes L_{((\mathbf{d},1),(-,0))_{Q_{\mathbf{f}}}}.$$

$L_{\mathbf{e}}$ ($\mathbf{e}\in\mathrm{Hom}(\mathbf{Z}^{Q_0},\mathbf{Z})$): simple lowest weight module for the generalised Kac–Moody algebra $\mathfrak{g}^{\mathrm{BPS}}_{\Pi_Q}$.

References

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