BPS Lie algebra of 2-Calabi–Yau categories and positivity of cuspidal polynomials of quivers

LUCIEN HENNECART
(joint work with Ben Davison and Sebastian Schlegel Mejia)

2-Calabi–Yau categories are ubiquitous in representation theory and algebraic geometry. They arise as the categories of

(1) Representations of the (deformed or not, additive or multiplicative) preprojective algebra Π_Q of a quiver Q, or more generally of 2-Calabi–Yau algebras,
(2) Representations of the (twisted or not) fundamental group algebra of a compact Riemann surface S,
(3) Semistable sheaves on (non-necessarily compact) symplectic surfaces.

This is a report on the preprints [4] and [5].

Setup. We let $\mathcal{A}$ be one of the categories defined above. We are here especially interested in the category $\mathcal{A} = \mathrm{Rep}(\Pi_Q)$ of finite dimensional representations of the preprojective algebra of a quiver Q. We refer to [4] for the general case. We let $(M,N)_{\mathcal{A}} := \sum_{j\in\mathbf{Z}}(-1)^j\mathrm{ext}^j(M,N)$ be the Euler form of $\mathcal{A}$.

Throughout, Q denotes a finite quiver, i.e. a pair of a set of vertices Q_0 and a set of arrows Q_1, both finite, along with two maps $s,t\colon Q_1\to Q_0$ assigning to an arrow its *source* and *target*. We form the *doubled quiver* $\overline{Q}=(Q_0,\overline{Q_1})$ by adding an arrow α^* to each arrow $\alpha\in Q_1$, with α^* given in the opposite orientation of α. The preprojective algebra is the quotient

$$\Pi_Q := \mathbf{C}\overline{Q}/\Big\langle \sum_{\alpha\in Q_1}[\alpha,\alpha^*]\Big\rangle.$$

Generalised Kac–Moody Lie algebra for a monoid with bilinear form. For a pair $\overline{M}=(M,(-,-))$ of a monoid with a bilinear form $(-,-)\colon M\times M\to\mathbf{Z}$, we define

$$\Sigma_{\overline{M}} := \Big\{m\in R^+_{\overline{M}} \mid \text{ for any nontrivial decomposition } m=\sum_{j=1}^r m_j, m_j\in M, \text{ one has } 2-(m,m) > \sum_{j=1}^r(2-(m_j,m_j))\Big\}$$

the set of primitive positive roots

and

$$\Phi^+_{\overline{M}} := \Sigma_{\overline{M}}\cup\{lm\colon l\geq 2, m\in\Sigma_{\overline{M}} \text{ with } (m,m)=0\}$$

the set of simple positive roots.

The Cartan matrix is $A_{\overline{M}} := ((m,n))_{m,n\in\Phi^+_{\overline{M}}}$. We assume that positive diagonal coefficients are equal to 2 and off-diagonal coefficients are nonpositive. For a

$\Phi^{+}_{\overline{M}}\times\mathbf{Z}$-vector space V, we define the Lie algebra $\mathfrak{n}_{\overline{M},V}$ as the Lie algebra generated by V with the relations

$$\begin{aligned} [v,w] &= 0 && \text{if } (\deg(v),\deg(w)) = 0 \\ \mathrm{ad}(v)^{1-(\deg(v),\deg(w))}(w) &= 0 && \text{if } (\deg(v),\deg(v)) = 2 \end{aligned}$$

for homogeneous $v, w \in V$, where $\deg\colon V \to \Phi^{+}_{\overline{M}}$.

The associative algebra generated by V with the same relations is canonically isomorphic to the enveloping algebra $\mathbf{U}(\mathfrak{n}_{\overline{M},V})$.

The BPS Lie algebra of 2 Calabi–Yau categories. We let $\mathfrak{M}_{\mathcal{A}}$ be the stack of objects of $\mathcal{A}$, $\mathcal{M}_{\mathcal{A}}$ be the moduli space of semisimple objects in $\mathcal{A}$ and $\mathtt{JH}\colon \mathfrak{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{A}}$ be the Jordan–Hölder map, sending an object of $\mathcal{A}$ to its semisimplification with respect to some Jordan–Hölder filtration. We let $\mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$ be the (Abelian) category of perverse sheaves on $\mathcal{M}_{\mathcal{A}}$. Using the monoid structure $\oplus\colon \mathcal{M}_{\mathcal{A}}^{\times 2} \to \mathcal{M}_{\mathcal{A}}$ given by the direct sum, we make $\mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$ a tensor category by defining the tensor product $\mathscr{F} \boxdot \mathscr{G} = \oplus_*(\mathscr{F} \boxtimes \mathscr{G})$. We let $M_{\mathcal{A}} := \pi_0(\mathcal{M}_{\mathcal{A}})$ be the monoid of connected components of $\mathcal{M}_{\mathcal{A}}$. We let $\mathcal{M}_{\mathcal{A},0}$ be the connected component of the zero object of $\mathcal{A}$. An algebra object in $\mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$ is a triple $(\mathscr{F} \in \mathrm{Perv}(\mathcal{M}_{\mathcal{A}}), m\colon \mathscr{F} \boxdot \mathscr{F} \to \mathscr{F}, \eta\colon \underline{\mathbf{Q}}_{\mathcal{M}_{\mathcal{A},0}} \to \mathscr{F})$ satisfying the usual axioms. Algebra objects in the category of bounded below constructible complex $\mathcal{D}^{+}_{\mathrm{c}}(\mathcal{M}_{\mathcal{A}})$ are defined in the same way.

Theorem 1 (Davison–H–Schlegel Mejia, 2022, [4, 5])**.**

1. *There is a cohomological Hall algebra product on the complex of constructible sheaves $\mathscr{A}_{\mathcal{A}} := \mathtt{JH}_*\mathbb{D}\underline{\mathbf{Q}}^{\mathrm{vir}}_{\mathfrak{M}_{\mathcal{A}}}$, making it an algebra object in $\mathcal{D}^{+}_{\mathrm{c}}(\mathcal{M}_{\mathcal{A}})$,*
2. *The constructible complex $\mathscr{A}_{\mathcal{A}}$ is semisimple and concentrated in nonnegative perverse degrees,*
3. *The degree 0 perverse cohomology ${}^{\mathfrak{p}}\mathcal{H}^0(\mathscr{A}_{\mathcal{A}})$ has an induced algebra structure in $\mathrm{Perv}(\mathcal{M}_{\mathcal{A}})$.*

The relative BPS algebra of $\mathcal{A}$ is defined as $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}} := {}^{\mathfrak{p}}\mathcal{H}^0(\mathscr{A}_{\mathcal{A}})$. The *absolute* BPS algebra is obtained by taking the derived global sections: $\mathrm{BPS}_{\mathcal{A},\mathrm{Alg}} := \mathrm{H}^*(\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}})$.

For $\mathcal{A} = \mathrm{Rep}(\Pi_Q)$, these results were proven in [2]. The proof in the generality exposed here relies on the neighbourhood theorem for 2-Calabi–Yau categories in [3].

Theorem 2 (Davison–H–Schlegel Mejia, 2023, [4, 5])**.** *The BPS algebra $\mathrm{BPS}_{\mathcal{A},\mathrm{Alg}}$ is isomorphic to the enveloping algebra of the generalised Kac–Moody Lie algebra associated to the pair $(M_{\mathcal{A}}, (-,-)_{\mathcal{A}})$ generated by*

$$\mathrm{IC}(\mathcal{M}_{\Phi^{+}_{\mathcal{A}}}) := \bigoplus_{a\in\Sigma_{\mathcal{A}}} \mathrm{IC}(\mathcal{M}_{\mathcal{A},a}) \oplus \bigoplus_{\substack{a\in\Sigma_{\mathcal{A}},(a,a)_{\mathcal{A}}=0\\ l\geq 2}} \mathrm{IC}(\mathcal{M}_{\mathcal{A},a}),$$

the intersection cohomology of some connected components of the moduli space of semisimple objects in $\mathcal{A}$ (note the specificity for isotropic roots).

Idea of the proof. This theorem is proven for the relative BPS algebra $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}$. First, using the neighbourhood theorem of [3], we show that it suffices to prove this theorem for $\mathcal{A} = \mathrm{Rep}(\Pi_Q)$ for all quivers Q. By the neighbourhood theorem again, we prove the result for preprojective algebras by induction on the set of pairs $(Q, \mathbf{d})$ of a quiver Q and a dimension vector $\mathbf{d} \in \mathbf{N}^{Q_0}$ supported on the whole of Q. We take advantage of the fact that $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}$ is a semisimple perverse sheaf on $\mathcal{M}_{\mathcal{A}}$. We then rely on one of the main theorems of [6] which gives an explicit and combinatorial description of the top CoHA of the strictly seminilpotent stack. □

At this point, one may define the relative BPS Lie algebra of $\mathcal{A}$ as the sub-Lie algebra of $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}$ generated by $\mathcal{IC}(\mathcal{M}_{\Phi^+_{\mathcal{A}}})$. The absolute BPS Lie algebra is $\mathrm{BPS}_{\mathcal{A},\mathrm{Lie}} \coloneqq \mathrm{H}^*(\mathcal{BPS}_{\mathcal{A},\mathrm{Lie}})$.

When $\mathcal{A}$ is the category of representations of a 2-Calabi–Yau algebra A, there is an other approach for defining the BPS Lie algebra using the critical cohomological Hall algebra associated to the 3-Calabi–Yau completion of A. In [4], we prove that both definitions lead to canonically isomorphic Lie algebras.

Corollary 3. *The BPS Lie algebra is isomorphic to the generalised Kac–Moody Lie algebra associated to the pair* $(\pi_0(\mathcal{M}_{\mathcal{A}}), (-,-)_{\mathcal{A}})$ *generated by* $\mathrm{IC}(\mathcal{M}_{\Phi^+_{\mathcal{A}}})$.

We let $A_{Q,\mathbf{d}}(q) \in \mathbf{N}[q]$, $\mathbf{d} \in \mathbf{N}^{Q_0}$ be Kac polynomials of the quiver Q.

Theorem 4 (Davison, [2])**.** *For* $\mathcal{A} = \mathrm{Rep}(\Pi_Q)$*, the character of the BPS Lie algebra is given by*

$$\mathrm{ch}(\mathrm{BPS}_{\Pi_Q,\mathrm{Lie}}) = \sum_{\mathbf{d} \in \mathbf{N}^{Q_0}} A_{Q,\mathbf{d}}(q^{-2}) z^{\mathbf{d}}.$$

Constructible Hall algebra and cuspidal polynomials. We let $\mathrm{Rep}(Q, \mathbf{F}_q)$ be category of representations of Q over the finite field with q elements $\mathbf{F}_q$. We denote by $\langle -,- \rangle_Q$ its Euler form. The *constructible* Hall algebra of Q is the space

$$H_{Q,\mathbf{F}_q} \coloneqq \mathrm{Fun}_{\mathrm{c}}(\mathrm{Rep}(Q, \mathbf{F}_q)/\!\sim, \mathbf{C})$$

of finitely supported functions on the set of isomorphism classes of representations of Q over $\mathbf{F}_q$. The algebra structure comes from the extension structure of the category $\mathrm{Rep}(Q, \mathbf{F}_q)$ and is given by some convolution product:

$$(f \star g)([R]) \coloneqq \sum_{N \subset R} q^{\frac{1}{2}\langle [R/N],[N] \rangle_Q} f([R/N]) g([N]),$$

Dually, a twisted coproduct Δ can be defined:

$$\Delta(f)([M],[N]) = \frac{q^{-\frac{1}{2}\langle M,N \rangle_Q}}{|\mathrm{Ext}^1_Q(M,N)|} \sum_{\xi \in \mathrm{Ext}^1(M,N)} f([X_\xi])$$

where X_ξ is the middle term of the short exact sequence determined by ξ.

The character of $H_{Q,\mathbf{F}_q}$ is given by the formulas with plethystic exponentials

$$\mathrm{ch}(H_{Q,\mathbf{F}_q}) \coloneqq \sum_{\mathbf{d}\in\mathbf{N}^{Q_0}} M_{Q,\mathbf{d}}(q)z^{\mathbf{d}} = \mathrm{Exp}_z\left(\sum_{\mathbf{d}\in\mathbf{N}^{Q_0}} I_{Q,\mathbf{d}}(q)z^{\mathbf{d}}\right)$$
$$= \mathrm{Exp}_{z,q}\left(\sum_{\mathbf{d}\in\mathbf{N}^{Q_0}} A_{Q,\mathbf{d}}(q)z^{\mathbf{d}}\right)$$

where the polynomials $M_{Q,\mathbf{d}}(q)$ (resp. $I_{Q,\mathbf{d}}(q)$, resp. $A_{Q,\mathbf{d}}(q)$) count all (resp. indecomposable, resp. absolutely indecomposable) $\mathbf{d}$-dimensional representations of Q over $\mathbf{F}_q$.

The space of *cuspidal functions* is the space of primitive elements for the coproduct Δ: $H^{\mathrm{cusp}}_{Q,\mathbf{F}_q} = \bigoplus_{\mathbf{d}\in\mathbf{N}^{Q_0}} H^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}]$, $H^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}] \coloneqq \{f \in H_{Q,\mathbf{F}_q}[\mathbf{d}] \mid \Delta(f) = f\otimes 1 + 1\otimes f\}$. Bozec and Schiffmann proved in [1] that the functions $C_{Q,\mathbf{d}}(q) \coloneqq \dim_{\mathbf{C}} H^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}]$ are polynomials in q. They conjectured that these polynomials have nonnegative coefficients for $\mathbf{d}\in\Sigma_{\Pi_Q}$.

Theorem 5 (Davison–H–Schlegel Mejia, 2023, [4, 5])**.** *For $\mathbf{d}\in\Sigma_{\Pi_Q}$, $C_{Q,\mathbf{d}}(q)\in\mathbf{N}[q]$. Furthermore, $C_{Q,\mathbf{d}}(q) = \mathrm{IP}(\mathcal{M}_{\Pi_Q,\mathbf{d}})(q^{-\frac{1}{2}})$ (intersection Poincaré polynomial).*

The proof of Theorem 5 relies on the interpretation of *absolutely* cuspidal polynomials as the $\mathbf{N}^{Q_0}\times\mathbf{Z}$-graded multiplicity of the space of simple positive roots of a $\mathbf{N}^{Q_0}\times\mathbf{Z}$-graded generalised Kac–Moody algebra having the generating series of Kac polynomials as character, [1]. Theorem 5 is then deduced from Corollary 3 and Theorem 4. Theorem 5 also provides qualitative information on the cuspidal polynomials: $C_{Q,\mathbf{d}}(q)$ is monic and of degree $1-\langle\mathbf{d},\mathbf{d}\rangle_Q$ for $\mathbf{d}\in\Sigma_{\Pi_Q}$.

References

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