Cuspidal functions and Lusztig sheaves for affine quivers

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I - Hall algebra (of a quiver)

II - Representation theory of affine quivers

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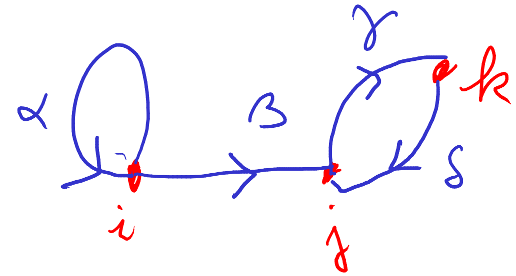
I - Hall algebra of a quiver

Hall algebra (Ringel, '90s)

$Q = (\underbrace{I}_{\text{vertices}}, \underbrace{\Omega}_{\text{arrows}})$ quiver



$I = \{i, j, k\}$

$\Omega = \{\alpha, \beta, \gamma, \delta\}$

$\mathbb{F}_q$ finite field

$$\boxed{H_{Q,\mathbb{F}_q} := \bigoplus_{[M] \in \mathrm{Rep}_Q(\mathbb{F}_q)/\mathrm{isom}} \mathbb{C}\cdot[M]}$$

$$= \mathrm{Fun}_c\left(\mathrm{Rep}_Q(\mathbb{F}_q)/\mathrm{isom}, \mathbb{C}\right)$$ finitely supported functions

Endow $H_{Q,\mathbb{F}_q}$ with the structure of (twisted) Hopf algebra.

Product: $$f * g([R]) = \sum_{\substack{M \subset R \\ \text{subobject}}} q^{\frac{1}{2}\langle R/M, M\rangle} f([R/M])\, g([M])$$ "convolution product"

Coproduct : $\Delta : H_{Q,\mathbb{F}_q} \longrightarrow H_{Q,\mathbb{F}_q} \otimes H_{Q,\mathbb{F}_q} \simeq \mathrm{Fun}_c(\mathrm{Rep}_Q(\mathbb{F}_q)/_\sim \times \mathrm{Rep}_Q(\mathbb{F}_q)/_\sim, \mathbb{C})$

$$\Delta(f)([N],[M]) = \frac{q^{-1/2\langle N,M\rangle}}{|\mathrm{Ext}^1(N,M)|} \sum_{\substack{0\to M\to R\to N\to 0\\ \in \mathrm{Ext}^1(N,M)}} f([R])$$

[counit, antipode: not relevant to us.]

Thm (Ringel, Green) Q without loops.

$\mathfrak{g}_Q = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ Kac-Moody algebra

$$\varphi : U_{\sqrt{q}}(\mathfrak{n}_+) \hookrightarrow H_{Q,\mathbb{F}_q}$$
$$E_i \longmapsto [S_i]$$

extends to an injective morphism of bialgebras.

- S_i = 1 dimensional rep. of Q concentrated at vertex $i \in I$
- $U_{\sqrt{q}}(\mathfrak{n}_+)$ = positive part of the quantum group specialized at $\sqrt{q}$

$= \mathbb{C}\langle E_i, i \in I\rangle /$ quantum Serre relations.

Facts: • φ is an isomorphism iff Q is of finite type

• The image of φ : $\mathrm{Im}\,\varphi =: H^{sph}_{Q,\mathbb{F}_q}$ is called the spherical Hall algebra of Q.

Questions: • Structure of $H_{Q,\mathbb{F}_q}$?
• Character of $H_{Q,\mathbb{F}_q}$?

Character of $H_{Q,\mathbb{F}_q}$

$Q = (I, \Omega)$ quiver

Kac polynomials:

$$M_{Q,d}(q) = \#\left(\mathrm{Rep}_Q(\mathbb{F}_q)[d] / \mathrm{isom}\right) \in \mathbb{N}[q]$$

$$I_{Q,d}(q) = \#\left(\mathrm{Rep}_Q^{\mathrm{indec}}(\mathbb{F}_q)[d] / \mathrm{isom}\right) \in \mathbb{Q}[q]$$

$$A_{Q,d}(q) = \#\left(\mathrm{Rep}_Q^{\mathrm{abs.indec}}(\mathbb{F}_q) / \mathrm{isom}\right) \in \mathbb{N}[q]$$

[Kac, Hausel–Letellier–Rodriguez-Villegas]

examples of Kac polynomials :

- for finite type ADE quivers, Gabriel's theorem $\Rightarrow$

$$A_{Q,d}(q) = \begin{cases} 1 & \text{if } d \in \mathbb{N}^I,\ \langle d, d\rangle = 1 \left(\begin{array}{c} d \text{ is a} \\ \text{positive root}\end{array}\right) \\ 0 & \text{else} \end{cases}$$

- for affine quivers :

$$A_{Q,d}(q) = \begin{cases} 1 & \text{if } d \in \mathbb{N}^I,\ \langle d, d\rangle = 1 \\ q + n_0 & \text{if } d \in \mathbb{N}^I,\ \langle d, d\rangle = 0 \\ 0 & \text{else} \end{cases}$$

- $S_g =$  g loops $\quad A_{S_g,1}(q) = q^g \ ; \ A_{S_g,2}(q) = q^{2g-1} \cdot \dfrac{1-q^{2g}}{1-q^2}, \ \ldots$

Kac polynomials enjoy remarkable combinatorial properties, but this is not the subject of today's talk.

Back to the character of the Hall algebra

$$\sum_{d\in\mathbb{N}^I} \dim H_{Q,\mathbb{F}_q}[d]\, z^d = \sum M_{Q,d}(q) z^d$$

$$\underset{\text{Krull-Schmidt}}{=} \mathrm{Exp}_z\left(\sum_{d\in\mathbb{N}^I} I_{Q,d}(q) z^d\right) \quad (*)$$

$$\underset{\substack{\text{Galois descent arguments}\\ \text{for quiver representations}}}{=} \mathrm{Exp}_{q,z}\left(\sum_{d\in\mathbb{N}^I} A_{Q,d}(q) z^d\right) \quad (**)$$

plethystic exponential

$(*)$ = character of the enveloping algebra of a $\mathbb{N}^I$-graded Lie $\tilde{\mathfrak{g}}_q$ algebra with character $\sum_{d\in\mathbb{N}^I} I_{Q,d}(q) z^d$

$(**)$ character of the enveloping algebra of a $\mathbb{N}\times\mathbb{N}^I$-graded Lie $\tilde{\mathfrak{g}}$ algebra with character $\sum_{d\in\mathbb{N}^I} A_{Q,d}(q) z^d$.

Do $\tilde{\mathfrak{g}}_q, \tilde{\mathfrak{g}}$ exist?

More precise question: Can we find

$\tilde{\mathfrak{g}}_q$ Borcherds Lie algebra s.t. $\mathrm{ch}\, \mathcal{U}(\tilde{\mathfrak{g}}_q^+) = (*)$? (1)

$\tilde{\mathfrak{g}}$ $\mathbb{N}\times\mathbb{N}^I$-graded Borcherds Lie algebra s.t. $\mathcal{U}(\tilde{\mathfrak{g}}^+) = (**)$? (2)

(1) Yes

(2) Hopefully

$\tilde{\mathfrak{g}}$ should be some "$q=1$" specialization of $\tilde{\mathfrak{g}}_q$.

Not easy since the number of generators of $\tilde{\mathfrak{g}}_q$ depends on q — below.

Structure of $H_{Q,\mathbb{F}_q}$:

$$H^{\mathrm{cusp}}_{Q,\mathbb{F}_q} = \{ f \in H_{Q,\mathbb{F}_q} \mid \Delta(f) = f\otimes 1 + 1\otimes f \}$$ space of cuspidal functions

Choose $(f_j)_{j\in J}$ homogeneous basis of $H^{\mathrm{cusp}}_{Q,\mathbb{F}_q}$.

$\langle -,- \rangle$ = Euler form of the quiver

$a_{ij} = \langle \dim f_i, \dim f_j \rangle$

Thm (Sevenhant - Van den Bergh).

$H_{Q,\mathbb{F}_q}$ is isomorphic to the associative $\mathbb{C}$-algebra generated by $(f_j)_{j\in J}$ with the relations

$$\cdot\ a_{ij} = 0 \Rightarrow [f_i, f_j] = 0$$

$$\cdot\ a_{ii} = 2 \Rightarrow \sum_{l=0}^{1-a_{ij}} (-1)^l \begin{Bmatrix} 1-a_{ij} \\ l \end{Bmatrix} f_i^l f_j f_i^{1-a_{ij}-l} = 0$$

quantum Serre relations

Consequence : $\cdot$ $\tilde{\mathfrak{g}}_q$ is the Borcherds Lie algebra with Cartan datum $(a_{ij})_{i,j}$.

$\cdot$ What about $\tilde{\mathfrak{g}}$? Not clear yet since $\dim H^{usp}_{Q,\mathbb{F}_q}[d]$ depends on q.

Fact / Theorem (Bozec-Schiffmann)

- $\dim H^{usp}_{Q,\mathbb{F}_q}[d]$ is a polynomial in q $=: C_{Q,d}(q)$
- If we interpret ✶✶ as the character of the enveloping algebra of a Borcherds Lie algebra, $\tilde{\mathfrak{g}}$, then the graded multiplicity of simple roots of $\tilde{\mathfrak{g}}$ are given by polynomials $C^{abs}_{Q,d}(q) \in \mathbb{Z}[q]$.

Conjecture: $C^{abs}_{Q,d}(q) \in \mathbb{N}[q]$. ($\Rightarrow$ existence of $\tilde{\mathfrak{g}}$)

Finite type quivers. $\mathfrak{g}_Q$ is a simple Lie algebra.

$U_{\sqrt{q}}(\mathfrak{n}^+) \longrightarrow H_{Q,\mathbb{F}_q}$ is an isomorphism

$$\Rightarrow \quad C_{Q,d}(q) = C^{abs}_{Q,d}(q) = \begin{cases} 1 & \text{if } d \in \{e_i, i \in I\} \\ 0 & \text{else} \end{cases}$$

cuspidal functions = characteristic functions of simple representations

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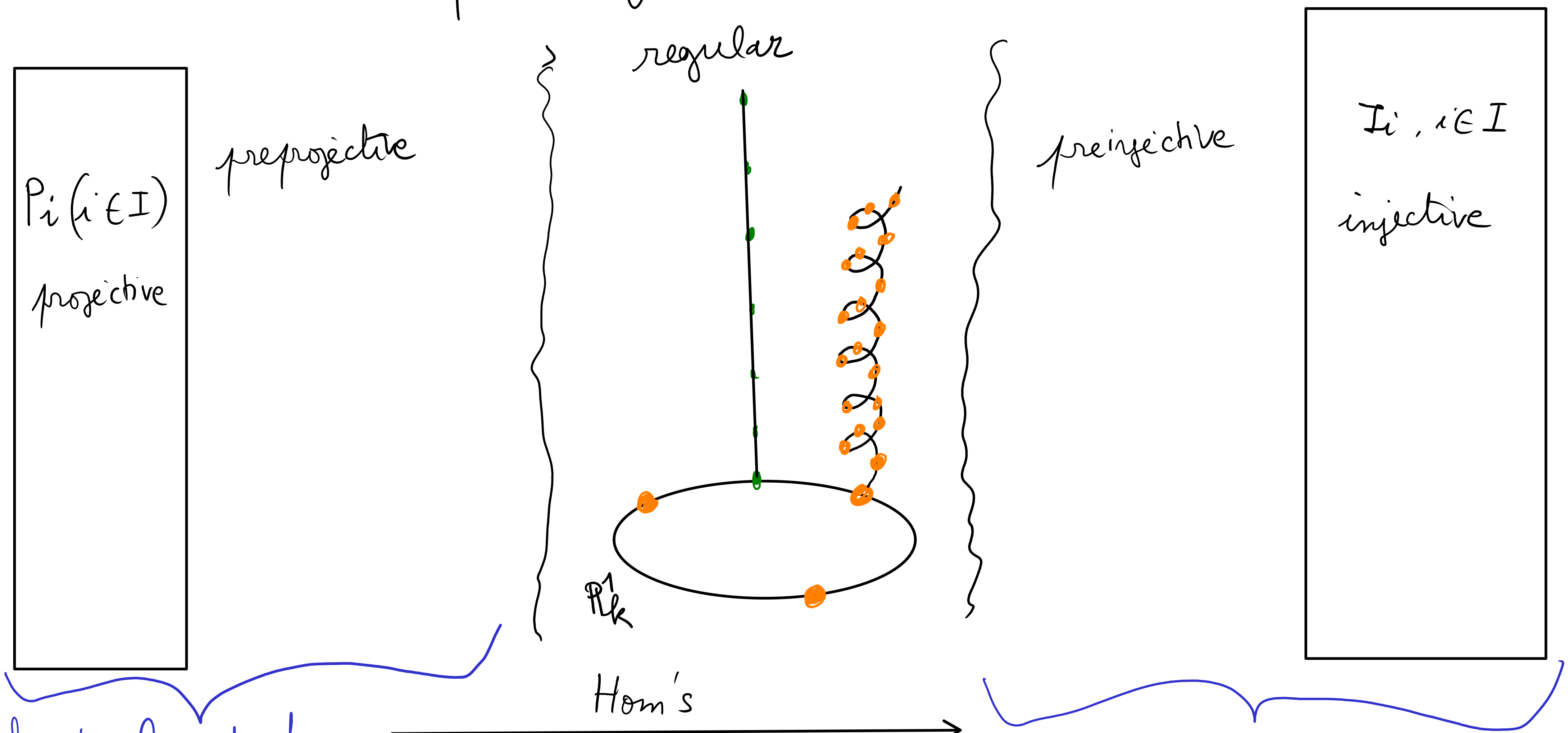
II- Representation theory of affine quivers.

Q affine ADE quiver. $\qquad$ k arbitrary field.

Auslander-Reiten quiver of Q



$$\xrightarrow{\quad \text{Hom's} \quad}$$

$$\xleftarrow[\quad \text{Ext's} \quad]{}$$

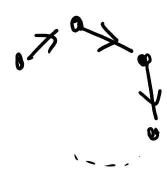
combinatorial part of the category: independent of k

- closed point of $\mathbb{P}^1_k$
- special point of degree 1.

$\mathrm{Rep}^{reg}_{Q}(k) \subset \mathrm{Rep}_{Q}(k)$ subcat. of regular representations.

Ringel: $\mathrm{Rep}^{reg}_{Q}(k) = \prod_{x \in |\mathbb{P}^1_k|} C_x$ $\quad (\Delta)$ $\qquad J =$  Jordan quiver

where $C_x \simeq \mathrm{Rep}^{nil}_{J}(k_x)$ except for a finite number of points $x \in |\mathbb{P}^1_k|$ of degree one, for which $C_x \simeq \mathrm{Rep}^{nil}_{C_p}(k)$,

$C_p =$  cyclic quiver of length $p = p_x$

Computation of cuspidal functions of affine quivers

1. consider the regular Hall algebra $H^{reg}_{Q,\mathbb{F}_q}$.
2. Compute $H^{reg,cusp}_{Q,\mathbb{F}_q}$: easy thanks to (Δ) if we know cuspidal functions of $H^{nil}_{J,\mathbb{F}_q}$ (nilpotent Hall algebra) and $H^{nil}_{C_p,\mathbb{F}_q}$ ($\forall p \geq 2$)
3. δ = indivisible imaginary root of Q.

Show $\quad H^{cusp}_{Q,\mathbb{F}_q}[r\delta] \subset H^{reg,cusp}_{Q,\mathbb{F}_q}[r\delta] \quad (\forall r \geq 1)$.

[analyze the support of a cuspidal function].

(4) Show the codimension is 1: dimension of r.h.s and l.h.s are explicitly known.

(5) Find a nontrivial linear form on $H^{reg,cusp}_{Q,\mathbb{F}_q}[r\delta]$ vanishing on $H^{cusp}_{Q,\mathbb{F}_q}[r\delta]$.

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(2) Cuspidal functions of $H^{nil}_{J,\mathbb{F}_q}$

Ring of symmetric functions

isomorphism of bialgebras $\quad \varphi_q : H^{nil}_{J,\mathbb{F}_q} \longrightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$

$\Lambda = \mathbb{Z}[x_i : i \in \mathbb{N}]^{\mathfrak{S}_{\mathbb{N}}}$.

$$[I_{(1^r)}] \longmapsto q^{-\frac{r(r-1)}{2}} e_r$$

$$\| $$

$$[(O_{r,r})]$$

$$e_r = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

$$\varphi_q([I_\lambda]) = q^{-n(\lambda)} P_\lambda(x; q^{-1}) \qquad n(\lambda) = \sum_i (i-1)\lambda_i .$$

$\lambda \in \mathcal{P}$ = partitions

Hall-Littlewood symmetric function

primitive elements of Λ : $p_r = \sum_i x_i^r \quad (r \geqslant 1)$

$$\varphi_q^{-1}(p_r) = \sum_{|\lambda| = r} \phi_{\ell(\lambda)-1}(q)\,[I_\lambda] , \quad \left(\phi_m(t) = \prod_{i=1}^m (1-t^i) , m \in \mathbb{N}\right)$$

(3) Use that a representation $M = P \oplus R \oplus I$ has a unique filtration

$$0 \subseteq F_1 \subseteq F_2 \subseteq M$$

with $F_1 \simeq I_1$, $F_2/F_1 \simeq R$, $M/F_2 \simeq P$.

$\Rightarrow$ $f([M]) = 0$ if f is cuspidal, and $P \oplus I \neq 0$,

(4) Have explicit formulas.

(5) $\int : H_{Q,\mathbb{F}_q}^{\mathrm{reg,cusp}}[r\delta] \longrightarrow \mathbb{C}$ is such a linear form.

$$f \longmapsto \sum_{[M] \in \mathrm{Rep}_Q^{\mathrm{reg}}(\mathbb{F}_q)/\mathrm{isom}} f([M])$$

Lusztig sheaves and geometrization of the Hall algebra

$Q = (I, \Omega)$

$\mathcal{M}_{Q,d} = [E_d / G_d]$ moduli stack of d-dim^{al} reps of Q.

$$E_d = \bigoplus_{\alpha : i \to j \in \Omega} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

$\circlearrowright$

$$G_d = \prod_{i \in I} GL_{d_i}(\mathbb{C}).$$

$T^*\mathcal{M}_{Q,d} = [\mu_d^{-1}(0) / G_d]$ moduli stack of representations of the preprojective algebra

$$T^*\mathcal{M}_{Q,d} \xrightarrow{p} 0 \in \mu_d^{-1}(0) /\!/ G_d$$

$\mu_d : T^* E_d \longrightarrow \mathfrak{gl}_d$ moment map.

$T^* E_d = E_{\overline{Q}, d}$

$$(x, x^*) \longmapsto \sum_{\alpha \in \Omega} [x_\alpha, x_\alpha^*]$$

$\Lambda_d = p^{-1}(0) \subset T^* \mathcal{M}_{Q,d}$ Lusztig nilpotent stack.

$\mathrm{Perv}(\mathcal{M}_{Q,d}, \Lambda_d) \subset \underbrace{D^b_c(\mathcal{M}_{Q,d}, \Lambda_d)}$

category of perverse sheaves whose singular support is a substack of Λ_d

constructible complexes whose sing. support is $\subset \Lambda_d$.

$$\begin{array}{ccccc}
 & & & & H_{top}(\Lambda_d) \\
 & & & & \wr\wr \\
K_0(\mathrm{Perv}(\mathcal{M}_{Q,d}, \Lambda_d)) & = & K_0(D^b_c(\mathcal{M}_{Q,d}, \Lambda_d)) & \xrightarrow{CC} & \mathbb{Z}[\mathrm{Irr}(\Lambda_d)] \\
? \, \| & & \cup & & \\
 & & K_0(\mathcal{P}_d) & \xrightarrow[\sim]{CC} & \mathbb{Z}[\mathrm{Irr}(\Lambda_d)]
\end{array}$$

Kashiwara - Saito.

$\mathcal{P}_d$ = Lusztig perverse sheaves on $\mathcal{M}_{Q,d}$.

$\bigoplus_{d \in \mathbb{N}^I} K_0(\mathrm{Perv}(\mathcal{M}_{Q,d}, \Lambda_d))$ has the induction product of Lusztig restriction

$\|$

$K_0(\mathcal{P})$

$(K_0(\mathcal{P}), \mathrm{ind}, \mathrm{res}) \simeq U^{\mathbb{Z}}(\mathfrak{n}_Q^+)$ $\qquad$ $\mathfrak{g}_Q = \mathfrak{n}_Q^- \oplus \mathfrak{h} \oplus \mathfrak{n}_Q^+$

$\bigoplus_{d \in \mathbb{N}^I} \mathbb{Z}[\mathrm{Irr}\, \Lambda_d]$ has the CoHA product [Schiffmann-Vasserot]

CC is an algebra (and coalgebra) map.

$\square$ is true • for finite type quivers : use that there is a finite number of G_d orbits in E_d

• for cyclic quivers by a Fourier-transform argument.

• for other affine quivers : also true, looking closely at the geometry of $\mathcal{M}_{Q,d}$ + explicit description of

(H2020)

$\mathcal{P}_d$ for affine quivers (Lusztig, Li-Lin).

Take $\mathcal{F} \in \mathrm{Perv}(\mathcal{M}_{Q,d}, \Lambda_d)$ a simple perverse sheaf

Want to show $\mathcal{F} \in \mathcal{P}_d$.

<u>Steps of the proof</u> : ① $\mathcal{M}_{Q,d} = \bigsqcup_{d_P+d_R+d_I=d} \mathcal{M}_{Q,d_P,d_R,d_I}$ stratification

One of the strata intersects $\mathrm{supp}\,\mathcal{F}$ densely.

$$\mathcal{M}_{Q,d_P,d_R,d_I} \longrightarrow \mathcal{M}^P_{Q,d_P} \times \mathcal{M}^R_{Q,d_R} \times \mathcal{M}^I_{Q,d_I}$$

"stack bundle" $\simeq$ fiber bundle

Reduce to the cases where supp $\mathcal{F}$ intersects densely the preprojective, preinjective or regular loci.

2. • preprojective and preinjective cases: easy, finite number of orbits

• regular case: study the geometry of $\mathcal{M}^{reg}_{Q,d}$ and Λ^{reg}_d.

$$\begin{tikzcd} \Lambda^{reg}_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Lambda_d \arrow[d] \\ \mathcal{M}^{reg}_{Q,d} \arrow[r, phantom, "\subset"] & \mathcal{M}_{Q,d} \end{tikzcd}$$

3. If $d \neq r\delta$ ($r \geq 1$), $\mathcal{M}^{reg}_{Q,d} = \emptyset$.

An open substack of $\mathcal{M}^{reg}_{Q,r\delta}$ is isomorphic to $S^r(\mathbb{P}^1 \setminus D / \mathbb{G}_m) \setminus \Delta$

"special points"

diagonal

trivial action of $\mathbb{G}_m$

Problem: need to know what happens on the boundary.

$\to$ describe a neighbourhood of non-homogeneous tubes in $\mathcal{M}^{reg}_{Q,d}$.

$$\begin{tikzcd}
\tilde{\Lambda}_{C_p, r\delta} \arrow[d] & \Lambda_U \arrow[l] \arrow[r] \arrow[d] & \Lambda_d^{reg} \arrow[d] \\
\mathcal{M}_{C_p, r\delta} \supset & U \subset & \mathcal{M}^{reg}_{Q, d=r\delta} \supset T
\end{tikzcd}$$

T = non homogeneous tube of period p

s.t. $\mathbb{C}^* U = \mathcal{M}_{C_p, r\delta}$

$\Rightarrow$ any $\mathbb{C}^*$- equivariant perverse sheaf on $\mathcal{M}_{C_p, r\delta}$ is determined by its restriction to U.

$\tilde{\Lambda}_{C_p, r\delta}$ is bigger than the Lusztig nilpotent stack for the cyclic quiver C_p.

4) Describe
- $\tilde{\Lambda}_{C_p, r\delta}$ explicitly,
- $\mathrm{Perv}(\mathcal{M}_{C_p, r\delta}, \tilde{\Lambda}_{C_p, r\delta})$.

5) Conclude.

What about general quivers (possibly w/ loops?)

$\to$ partial results.

- quivers S_g  : OK. use the seminilpotent stacks.

Extension of the question to curves:

Lusztig sheaves $\mathcal{P}$ ⇝ spherical Eisenstein perverse sheaves $\mathcal{P}$

Lusztig nilpotent stack Λ ⇝ global nilpotent cone $\mathcal{N}$

- We have an algebra morphism

$$CC : \underbrace{K_0(\mathcal{P})}_{\text{induction product}} \longrightarrow \mathbb{C}[\underbrace{\mathrm{Irr}}_{\text{CoHA product}} \mathcal{N}]$$

- Can show it is surjective
- injectivity is tricky.
  Known for curves of genus ≤ 1 only. (H2021)
- coalgebra morphism?

all this will be joint project with Ben Davison and Olivier Schiffmann.

Thank you for your attention!