The Hall algebra of curves and quivers: cuspidal functions, perverse sheaves and Kac polynomials.

Philip Hall – british mathematician ~ 1950 "The algebra of partitions"
expanded, reproved results of Ernst Steinitz 1901
"Zur Theorie der abelschen Gruppen"

finite abelian p groups ; $G = \prod_{i=1}^{N} \mathbb{Z}/p^{\lambda_i} =: G_\lambda$

$\lambda = (\lambda_1 \geq \dots \geq \lambda_N)$ partition.

$\left[\text{such groups}/\sim \quad \overset{1:1}{\longleftrightarrow} \quad \mathcal{P} = \text{partitions} \right]$

$H_p := \bigoplus_{\lambda \in \mathcal{P}} \mathbb{Z}[G_\lambda]$ can be endowed w/ an algebra structure

$$[G_\lambda]\cdot[G_\mu] = \sum_{\nu \in \mathcal{P}} a^{\nu}_{\lambda,\mu}(p)[G_\nu]$$

$$a^{\nu}_{\lambda,\mu}(p) = \#\{ H \subset G_\nu \mid H \simeq G_\mu \ \& \ G_\nu/H \simeq G_\lambda \}$$

Hall numbers

H_p : associative, commutative algebra.

Thm (Hall) ① $H_p \underset{\mathbb{Z}}{\otimes} \mathbb{C} \simeq \Lambda_{\mathbb{C}}$ (MacDonald Ring of symmetric functions)
algebra isomorphism.

$\Lambda_{\mathbb{Z}} = \mathbb{Z}[x_i : i \geq 1]^{\mathfrak{S}_\infty}$.

② $a^{\nu}_{\lambda,\mu}(p) \in \mathbb{Z}[p]$ is a polynomial in p.

H_p = "algèbre de Hall classique" [p-groupes abéliens finis $\leftrightarrow$ représentations finies de l'anneau des entiers p-adiques]

Ringel, Green : replace finite abelian p-groups by representations of a quiver Q over a finite field $\mathbb{F}_q$.

Striking result : construction of quantum groups of Drinfeld-Jimbo

$Q = (I, \Omega)$ quiver (finite oriented graph)

I : vertices, Ω : arrows

$\mathrm{Rep}_Q(\mathbb{F}_q)$ rep. of Q over the finite field $\mathbb{F}_q$

$$H_{Q,\mathbb{F}_q} := \bigoplus_{[M] \in \mathrm{ob}(\mathrm{Rep}_Q(\mathbb{F}_q))/\sim} \mathbb{C}[M]$$

+ algebra, $\boxed{\text{coalgebra}}$ structures

$$[M]\cdot[N] = \sum_{[R] \in \mathrm{ob}(\mathrm{Rep}_Q(\mathbb{F}_q))/\sim} a^{R}_{M,N}\,[R]$$

Δ comultiplication dual to the multiplication

Q has no loops

$\mathfrak{g}_Q$ Kac Moody algebra

$Q = \bullet$ $\qquad \mathfrak{g}_Q = \mathfrak{sl}_2$

$Q = \bullet\!-\!\bullet \cdots \bullet\!-\!\bullet$ (n vertices) $\qquad \mathfrak{g}_Q = \mathfrak{sl}_n$

$Q = \bullet \rightrightarrows \bullet$ $\qquad \mathfrak{g}_Q = \widehat{\mathfrak{sl}_2}$ affine Lie algebra

⋮

$U(\mathfrak{g}_Q)$ enveloping algebra

$\downarrow$ 1-parameter deformation (Drinfeld, Jimbo)

$U_v(\mathfrak{g}_Q)$ quantum group

Thm (Ringel, Green) $\quad H_{Q,\mathbb{F}_q} \xleftarrow{\ \Phi\ } U_{\sqrt{q}}(\mathfrak{g}_Q^{+})$

The image of Φ is the "spherical" Hall algebra, is generated by $[s_i]$, $i \in I$

Φ is an isomorphism $\Leftrightarrow$ $\mathfrak{g}_Q$ is a semisimple Lie algebra.
$\Leftrightarrow$ Q is of finite type

<u>Question</u>: What is the structure of the whole Hall algebra $H_{Q,\mathbb{F}_q}$?

<u>Answer</u>: Sevenhant–Van den Bergh.

$$H^{cusp}_{Q,\mathbb{F}_q} := \{ f \in H_{Q,\mathbb{F}_q} \mid \Delta f = f \otimes 1 + 1 \otimes f \}$$ cuspidal functions

$(f_j)_{j \in J}$ homogeneous basis

$\deg f_j \in \mathbb{Z}^I$, $j \in J$.

$a_{jk} = (\deg f_j, \deg f_k)$ $\qquad (-,-)$ symmetrized Euler form of Q.

<u>Thm</u> (S-VdB) $H_{Q,\mathbb{F}_q} \simeq U_{\sqrt{q}}(\mathfrak{g}^+_{JB})$

$\mathfrak{g}^+_{JB}$ is the positive part of the Borcherds Lie algebra with Cartan matrix $(a_{jk})_{j,k \in J}$

<u>Question</u>: Can we say more about $H^{cusp}_{Q,\mathbb{F}_q} = \bigoplus_{d \in \mathbb{N}^I} H^{cusp}_{Q,\mathbb{F}_q}[d]$?

<u>Thm</u> (Bozec-Schiffmann)

$$\dim_{\mathbb{C}} H^{cusp}_{Q,\mathbb{F}_q}[d] \begin{cases} \neq 0 & \overset{\text{seulement si}}{\Rightarrow} (d, e_i) \leq 0 \ \forall i \in I, \ d \text{ is connected} \\ \in \mathbb{Q}[q] & \\ \in \mathbb{Z}[q] & \text{if } (d,d) < 0 \ \text{(hyperbolic root)} \end{cases}$$

finite type quivers: [Ringel's thm $\Rightarrow$] $H^{cusp}_{Q,\mathbb{F}_q} = \bigoplus_{i\in I} \mathbb{C}[S_i]$

Affine quivers:

Thm (H.) There exists a subalgebra $H_{Q,\mathbb{F}_q,\mathcal{R}}$, the regular Hall algebra, endowed w/ a coproduct $\Delta_{\mathcal{R}}$, such that for $r \geq 0$, δ the imaginary indivisible proot of Q,

$$H^{cusp}_{Q,\mathbb{F}_q}[r\delta] \underset{\text{codimension } 1}{\subset} H^{cusp}_{Q,\mathbb{F}_q,\mathcal{R}}[r\delta].$$

and a $\underset{\text{(canonical)}}{}$ linear form $\chi_{r\delta} : H^{cusp}_{Q,\mathbb{F}_q,\mathcal{R}}[r\delta] \longrightarrow \mathbb{C}$

whose kernel is $H^{cusp}_{Q,\mathbb{F}_q}[r\delta]$.

The algebra $H_{Q,\mathbb{F}_q,\mathcal{R}}$ comes from the representation theory of Q

Q (acyclic) quiver.

Auslander–Reiten theory [gives us a partition of]

$$\mathrm{Ind}(Q) = \{\text{indec. reps of } Q\}/\sim \; = \underbrace{\mathcal{P}}_{\text{ind preprojective reps}} \sqcup \underbrace{\mathcal{R}}_{\text{ind preinjective reps}} \sqcup \underbrace{\mathcal{I}}_{\text{postinjective reps}}$$

regular representations of Q: $\underset{\text{abelian subcategory if } Q \text{ is affine}}{\mathrm{Rep}^{\mathcal{R}}_Q(\mathbb{F}_q) \subset \mathrm{Rep}_Q(\mathbb{F}_q)}$.

$H^{\mathcal{R}}_{Q,\mathbb{F}_q}$ = Hall algebra of $Rep^{\mathcal{R}}_Q(\mathbb{F}_q)$

Thm (Ringel)

$$Rep^{\mathcal{R}}_Q(\mathbb{F}_q) \simeq \bigsqcup_{x\in|\mathbb{P}^1_{\mathbb{F}_q}|} C_x$$

where $C_x \simeq Rep^{nil}_{C}(\mathbb{F}_{q^{\deg(x)}})$ nilpotent representations of some cyclic quiver

→ the algebra $H^{\alpha}_{Q,\mathbb{F}_q,\mathcal{R}}$ is completely known

→ the primitive elements $H^{cusp}_{Q,\mathbb{F}_q,\mathcal{R}}$ are easily computable.

→ "fortuitous cancellation theorem" : $H^{cusp}_{Q,\mathbb{F}_q}[r\delta] \subset H^{cusp}_{Q,\mathbb{F}_q,\mathcal{R}}[r\delta]$
résultat technique

→ take $\chi_{r\delta}$ = integration against the orbifold measure of $Rep_Q(\mathbb{F}_q)$.

Next goal : towards a geometric notion of cuspidality

- Hall algebra = convolution algebra on the space of functions of $Rep_Q(\mathbb{F}_q)/\sim \longrightarrow \mathbb{C}$.

- Lusztig idea : consider perverse sheaves on the moduli stack of representation of Q

$$\mathcal{M}_Q = \bigsqcup_{d\in\mathbb{N}^I} \mathcal{M}_{Q,d}$$

k field.

$\mathcal{M}_{Q,d} = E_d / G_d$ quotient stack

$$E_d = \bigoplus_{\substack{\alpha \in \Omega \\ i \to j}} \mathrm{Hom}(k^{d_i}, k^{d_j}) \quad ; \quad G_d = \prod_{i \in I} GL_{d_i}.$$

Lusztig defines $\mathcal{Q} \subset D^b_c(\mathcal{M}_Q)$ category of semi-simple complexes, such that $K_\oplus(\mathcal{Q}) \simeq U_q^{\mathbb{Z}}(\mathfrak{g}_Q^+)$ integral form of the positive part of the quantum group.

$\mathcal{P} \subset \mathcal{Q}$ perverse sheaves give the canonical basis of $U_q^{\mathbb{Z}}(\mathfrak{g}_Q^+)$ defined combinatorially by Kashiwara.

Question: Intrinsic characterization of $\mathcal{Q}$?
Answer: singular support condition

$\Lambda \subset T^*\mathcal{M}_Q$ Lusztig nilpotent stack

Define $D^b_c(\mathcal{M}, \Lambda)$ = category of constructible complexes $\mathcal{F}$ on $\mathcal{M}$ s.t. $SS(\mathcal{F}) \subset \Lambda$.

Conjecture (Lusztig, Webster) The fully faithful functor $\mathcal{Q} \longrightarrow D^b_c(\mathcal{M}, \Lambda)$ induces an isomorphism $K_0(\mathcal{Q}) \simeq K_0(D^b_c(\mathcal{M}, \Lambda))$.

Thm (Lusztig) The conjecture holds for finite type quivers.

Thm (H.). ——— affine ———

Thm (H.). The conjecture holds for S_g = quivers for the appropriate notions of Lusztig sheaves and the appropriate nilpotent stack.

~~Strategy of proof for affine quivers:~~

- ~~Auslander-Reiten theory gives a stratification of $M_{Q,d}$~~
- ~~this stratification allows to describe explicitly~~
  ~~① the simple perverse sheaves of $\mathcal{P}$ (Lusztig, Li-Lin)~~
  ~~② the irreducible components of Λ (Ringel)~~
  ~~③ Study the perverse sheaves with nilpotent singular support and prove the conjecture for affine quivers (H.)~~

Lusztig complexes / perverse sheaves:

$Q=(I,\Omega)$ quiver

$d \in \mathbb{N}^I$ dim. vector

$\underline{d}=(d_1,\dots,d_s)\in(\mathbb{N}^I)^s$ s.t. $\sum_{i=1}^{s} d_i = d$. ;

$V := \mathbb{C}^d$ I-graded, d dim $\mathbb{C}$-v.s.

universal quiver flag-variety : $\widetilde{\mathcal{F}}_{\underline{d}}$ = stack of pairs $(x, F_\bullet)$,

$$0=F_0\subset F_1\subset\dots\subset F_s=V$$
$$\dim F_i/F_{i-1}=d_i$$
$$xF_i\subset F_i$$

$\widetilde{\mathcal{F}}_{\underline{d}}$ is smooth, $\widetilde{\mathcal{F}}_{\underline{d}} \xrightarrow{\pi_{\underline{d}}} M_{Q,d}$ is proper.

$$(x,F_\bullet)\longmapsto x$$

Decomposition thm (Beilinson-Bernstein-Deligne-Gabber):

$$\pi_{\underline{d}}\,\underline{\mathbb{C}}\in D^b_c(M_{Q,d},\mathbb{C})$$

is a semisimple complex.

$\underline{d}$ "discrete" if $\forall\, 1\le i\le s$, d_i is concentrated at one vertex

Lusztig categories $\mathcal{P}=\bigsqcup_{d\in\mathbb{N}^I}\mathcal{P}_d$ $\mathcal{P}_d$ = s.s perverse sheaves on $M_{Q,d}$ whose simple direct summands appear in some induction $(\pi_{\underline{d}})_*\,\underline{\mathbb{C}}$

$$\mathcal{Q} = \bigoplus_{j \in \mathbb{Z}} \mathcal{P}[j] \quad \subset \mathcal{D}^b_c(\mathcal{M}_Q, \mathbb{C}) \qquad \text{triangulated category}$$

Lusztig nilpotent stack

$$Q \rightsquigarrow \overline{Q} = (I, \Omega \sqcup \overline{\Omega})$$

doubled quiver

$$\bullet \xrightarrow{\alpha} \bullet \qquad \begin{tikzcd} \bullet \arrow[r, bend left, "\alpha"] & \bullet \arrow[l, bend left, "\alpha^*"] \end{tikzcd}$$

$\mathbb{C}\overline{Q}$ path algebra of $\overline{Q}$

$\cup\!\!\!\!\!\;|$

$$m = \sum_{\alpha \in \Omega} [\alpha, \alpha^*] \qquad \rightsquigarrow \Pi_Q = \mathbb{C}\overline{Q}/m \quad \text{preprojective algebra.}$$

$\mathcal{M}_{\Pi_Q}$ = stack of reps of Π_Q

$\underset{\text{Hamiltonian reduction}}{=} T^*\mathcal{M}_Q$

$\Lambda \subset \mathcal{M}_{\Pi_Q}$ [closed, conical, Lagrangian substack]

Lusztig nilpotent stack

Λ = nilpotent representations of Π_Q.

<u>Fact</u> (Lusztig): If $\mathcal{F} \in \mathcal{P}$, $SS(\mathcal{F}) \subset \Lambda$.

Lusztig conjecture: converse.

Strategy for affine quivers

① Auslander-Reiten theory [gives a stratification]

$$\mathcal{M}_{Q,d} = \bigsqcup_{d_I + d_R + d_P} \mathcal{M}_{d_I, d_R, d_P}.$$

since each rep M of Q has a canonical filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset M_3 = M$$

s.t. M_1 is preprojective

M_2/M_1 is regular

M_3/M_2 is postinjective.

(which (non-canonically) splits).

$\Rightarrow$

$$\begin{array}{c} \mathcal{M}_{d_I, d_R, d_P} \\ \downarrow p \quad \text{smooth, connected fibers.} \\ \mathcal{M}^I_{d_I} \times \mathcal{M}^R_{d_R} \times \mathcal{M}^P_{d_P} \end{array}$$

$\mathcal{M}^I_{d_I} \subset \mathcal{M}_{d_I}$

[loc. closed substack classifying d_I-dim post-injective reps of Q]

② reduce to p.sheaves on $\mathcal{M}^I_{d_I}$, $\mathcal{M}^R_{d_R}$, $\mathcal{M}^P_{d_P}$.

③ cases of $\mathcal{M}^I_{d_I}$, $\mathcal{M}^P_{d_P}$: [use that there are] finitely many orbits.

④ case of $\mathcal{M}^R_{d_R}$: [stratify this stack which] locally looks like $\mathfrak{gl}_n/GL_n$ except at finitely many points where it looks like $\mathcal{M}_{C_p, \tilde{d}}$ C_p cyclic quiver of length p.

⑤ [Reduce to the] cyclic quiver C_p & Springer theory for $\mathfrak{gl}_n$.

__Situation of curves:__ X smooth projective curve

The relation $\mathcal{Q} \leftrightarrow \Lambda$ has an analogue for curves

$\mathcal{Q}$ is replaced by "spherical Eisenstein complexes"

Λ is replaced by the global nilpotent cone

. Using similar methods, we can prove that the characteristic cycle map

$$CC : \widehat{K_0(\mathcal{Q})} \longrightarrow \widehat{\mathbb{Z}[\mathrm{Irr}\,\Lambda]}$$

between (appropriate completions) is an isomorphism. (H2021) when X is an elliptic curve,

and give and explicit description of (simple) perverse sheaves on $\mathcal{C}oh(X)$ whose singular support is contained in Λ.

- Non-trivial local systems on the curve prevent us from having a "microlocalisation" as for quivers.

In a slightly different direction: the study of Kac polynomials.

Kac (1980's) defined three families of polynomials:

$$M_{Q,d}(q) = \#\{\text{reps of } Q \text{ over } \mathbb{F}_q\}/\sim$$

$$I_{Q,d}(q) = \#\{\text{indecomposable reps of } Q \text{ over } \mathbb{F}_q\}/\sim$$

$$A_{Q,d}(q) = \#\{\text{abs. indec reps of } Q \text{ over } \mathbb{F}_q\}/\sim$$

+ series of conjectures for the A-family.

- $A_{Q,d}(q) \in \mathbb{N}[q]$ (Hausel–Letellier–Rodriguez-Villegas 2013)
- $A_{Q,d}(0) = \dim \mathfrak{g}_Q[d]$. (Hausel 2008?)

à part.

They give the character of the Hall algebra:

$$\operatorname{ch} H_{Q,\mathbb{F}_q} := \sum_{d \in \mathbb{N}^I} \dim H_{Q,\mathbb{F}_q}[d]\, z^d \in \mathbb{Q}[[z_i : i \in I]]$$

$$= \sum_{d \in \mathbb{N}^I} M_{Q,d}(q) \cdot z^d$$

$$= \operatorname{Exp}_z\left(\sum_{d \in \mathbb{N}^I} I_{Q,d}(q)\, z^d\right)$$

$$= \operatorname{Exp}_{z,q}\left(\sum_{d \in \mathbb{N}^I} A_{Q,d}(q)\, z^d\right)$$

plethystic exponentials.

and cuspidal polynomials $C_{Q,d}(q) = \dim H_{Q,\mathbb{F}_q}[d]$ are built recursively from them.

<u>Setup</u>: $Q = (I, \Omega)$ quiver

for $\underline{n} \in \mathbb{N}^{\Omega}$ $\to$ [new quiver] $Q_{\underline{n}}$ [obtained from Q by replacing each arrow $\alpha \in \Omega$ by n_α arrows.]

$\underline{n} = (n_\alpha)_{\alpha \in \Omega}$

sequence of polynomials $\left(A_{Q_{\underline{n}}, d}(q)\right)_{\underline{n} \in \mathbb{N}^{\Omega}}$.

<u>Thm</u>: As $\underline{n} \longrightarrow \underline{m} \in (\mathbb{N} \cup \{+\infty\})^{\Omega}$, the sequence $\left(A_{Q_{\underline{n}}, d}(q)\right)_{\underline{n}}$ converges to a power series in $\mathbb{N}[[q]]$ which is the power series expansion at $q=0$ of a rational fraction.

[<u>Rk</u>. If Q has loops, we need to take some renormalization to avoid the limit being 0.]

[This theorem is obtained as a corollary to the following structural result.]

<u>Theorem</u>:
$$A_{Q_{\underline{n}}, d}(q) = \frac{\sum_j q^{l_j(\underline{n})} P_j(q)}{Q(q)}$$

* $P_j(q), Q(q) \in \mathbb{Z}[q]$
* the roots of Q are roots of unity
* $l_j : \mathbb{Z}^I \longrightarrow \mathbb{Z}$ are affine functions with linear parts pairwise distinct
+ ... ensuring unicity of the decomposition.

[This theorem combined with computational techniques provides an efficient method to determine Kac polynomials.]

For example, if $Q = 1 \xrightarrow{\alpha} 2 \circlearrowleft \beta$

$$A_{Q_{\underline{n}}, d, 0} := \frac{A_{Q_{\underline{n}}, d}}{q^{1 + d_2(n_\beta - 1)}},$$

$$A_{Q_{\underline{n}},(1,2),0} = \frac{1+q-q^{n_\alpha}(1+q)-q^{2n_\beta}+q^{2(n_\alpha+n_\beta)}}{(1-q)(1-q^2)}$$

Proof of the theorem: Carefully inspect Hua's formula expressing $\sum_{d\in\mathbb{N}^I} A_{Q,d}(q)z^d$ and the plethystic exponential.

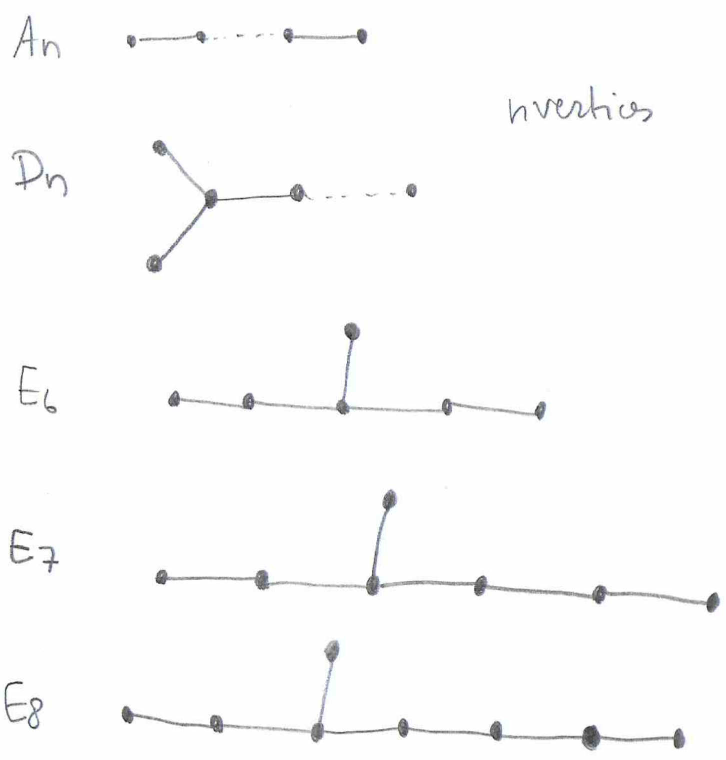
The unicity of the writing of Kac polynomials gives new invariants associated with the quiver Q:

- the affine functions l_j
- the polynomials $P_j(q)$.

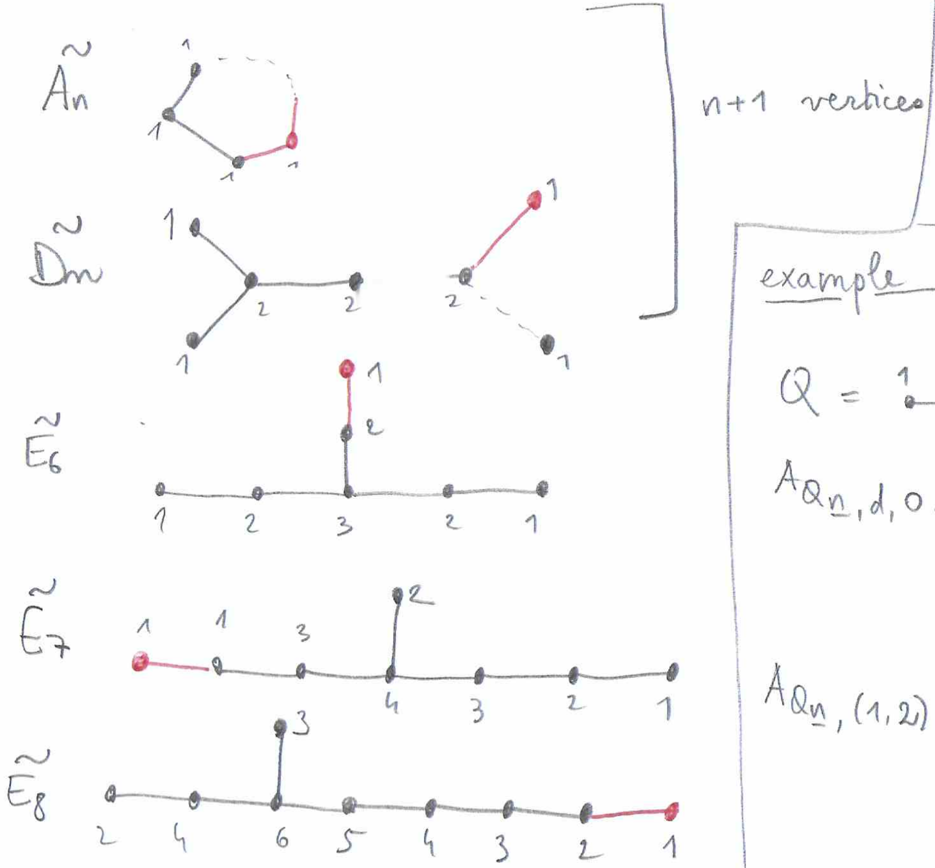
of which I know nothing yet.

Annexe

Finite type quivers



Affine quivers



example de polynôme de Kac

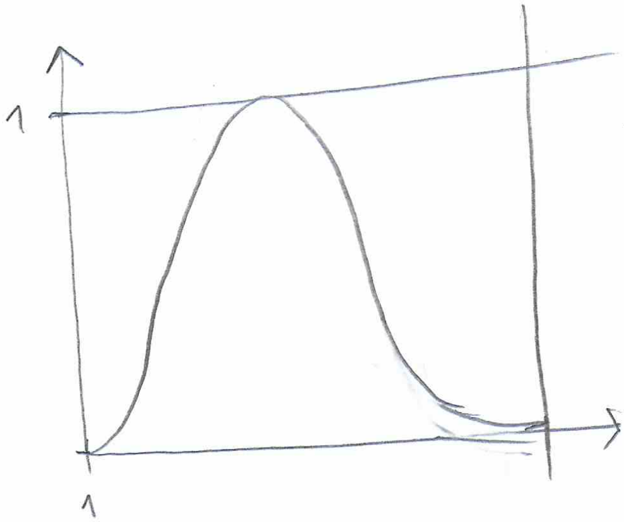
$$Q = 1 \xrightarrow{\alpha} 2 \circlearrowleft \beta$$

$$A_{Q_{\underline{n}},d,0} := \frac{A_{Q_{\underline{n}},d}}{q^{1+d_2(n_\beta - 1)}}$$

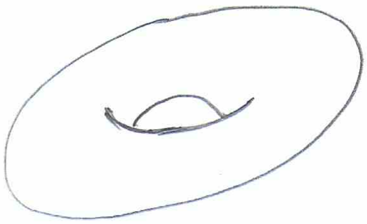
$$A_{Q_{\underline{n}},(1,2),0} = \frac{1+q-q^{n_\alpha}(1+q)-q^{2n_\beta}+q^{2(n_\alpha+n_\beta)}}{(1-q)(1-q^2)}$$

Hua's formula

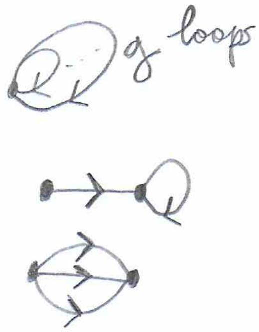
$$\sum_{d\in\mathbb{N}^I} A_{Q,d}(q)\, z^d = (q-1)\,\mathrm{Log}_{z,q}\left(\sum_{\pi=(\pi^i)_{i\in I}\in\mathcal{P}^I} \frac{\prod_{\alpha:i\to j\in\Omega} q^{\langle \pi^i,\pi^j\rangle}}{\prod_{i\in I} q^{\langle \pi^i,\pi^i\rangle} \prod_k \prod_{j=1}^{m_k(\pi^i)} (1-q^{-j})}\, z^{|\pi|} \right)$$



Airy distribution



Carquois sauvages (exemples)



Auslander–Reiten translates

(τ^-, τ) Adjunction
Serre

$$\mathrm{Ext}^1(M,N)^* \simeq \mathrm{Hom}(N, \tau M)$$
$$\simeq \mathrm{Hom}(\tau^- N, M)$$

kQ hereditary:

$$\tau M = \mathrm{Ext}^1_{kQ}(M, kQ)$$

$$\tau^- M =$$

$$\begin{array}{ccc} H^{nil}_{J,\mathbb{F}_q} \underset{q=p}{\simeq} H_p \otimes_{\mathbb{Z}} \mathbb{C} & \longrightarrow & \Lambda_{\mathbb{C}} \\ \rotatebox{90}{$\in$} & & \\ [G_\lambda] & \longmapsto & q^{-n(\lambda)} \underbrace{P_\lambda(x;q^{-1})}_{\substack{\text{Hall-Littlewood} \\ \text{symmetric function}}} \end{array}$$

$$n(\lambda) = \sum_i (i-1)\lambda_i .$$

$p_r \in \Lambda_{\mathbb{C}}$ $\qquad p_r = \sum_i x_i^r$

$$\tilde{p_r} = \sum_{|\lambda|=r} \phi_{\ell(\lambda)-1}(q)\,[G_\lambda]$$

$$\phi_m(q) = \prod_{i=1}^{m} (1-t^i)$$

<u>cuspidaux (nilpotents) carquois cycliques</u>

n sommets

$$A_k^n = \left\{ [M] : M \in \mathrm{Rep}^{nil}_{C_n}(\mathbb{F}_q)[\delta] \text{ with exactly } k+1 \text{ vanishing arrows} \right\}$$

$$f_{\delta_n} = \sum_{k=0}^{n-1} (1-q)^k \sum_{[M]\in A_k^n} [M]$$