Localization of equivariant cohomology

Goal: introduce the equivariant cohomology package:

- $H^*_G(X)$ equivariant cohomology
- (T torus) $H^*_T(X) \xrightarrow{\iota^*} H^*_T(X^T)$ is injective map of $H^*(pt)$-mod and becomes an isomorphism over $H^*(pt)[e(N^T_{X^T/X})^{-1}]$.
- integration formula (Gysin homomorphism) $f_*: H^*_T(X) \longrightarrow H^*_T(Y)$
- image theorem: description of the image of ι^*.

Prerequisites: the (singular) cohomology package, including

- Chern classes of vector bundles.
- Gysin homomorphism $f_*: H^*(X) \longrightarrow H^*(Y)$ for X, Y smooth and f proper.
- The self intersection formula $f^* f_* \in \mathrm{End}(H^*(X))$ is the multiplication by $e(N_{X/Y})$ for $Y \overset{f}{\hookrightarrow} X$ closed X, Y smooth.

[or more generally, of l.c.i.].

- cohomology of fibrations (Leray-Hirsch)
- Gysin sequence

$\Downarrow$

$$\begin{tikzcd} E \arrow[d] \\ X \end{tikzcd}$$

E (oriented) vector bundle over X,

$S(E)$ sphere bundle

$$\begin{tikzcd} H^{q-n}(S(E)) \arrow[r] & H^{q-n}(X) \arrow[r, "\text{cup w/ Euler class of } E", "\text{injective sometimes!}"'] & H^q(X) \arrow[r] & H^q(S(E)) \end{tikzcd}$$

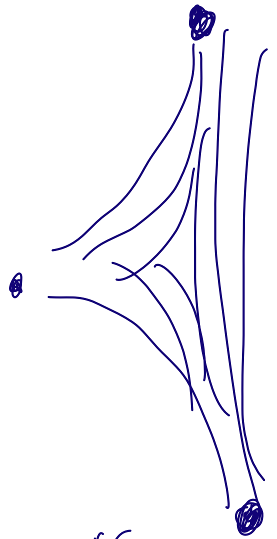
① Equivariant cohomology

G group
X a G-topological space) $\rightsquigarrow H^*_G(X)$ eq. coh.

algebro-topological invariant of $X \circlearrowleft G$.

Puts a the same level the representation theory of groups and the topology of spaces
[more true for equivariant K-theory]

It is a better object than the cohomology $\underline{H^*(X/G)}$ of the quotient space for what concerns algebro-geometric properties of $G \curvearrowright X$.



- $H^*(X/G)$ looses information on stabilizers of points
- how points flow.

The thing making $H^*(X/G)$ bad is that it not compatible with homotopy equivalences.

- In favourable situations, equivariant cohomology can be understood combinatorially (GKM, moment graph)

Definition $X \circlearrowleft G$

$$H^*_G(X) = H^*(\underbrace{X/^h G}_{\text{homotopy quotient of } X \text{ by } G})$$

is $H^*_G(pt)$-module $\Rightarrow$ coherent sheaf on $\operatorname{Spec} H^*_G(pt)$

A model for $X/^hG$ is $X \times^G E$ where

E is a contractible right free G-space;

$$X\times^G E = {}_G\backslash X \times E \quad \text{where}$$

$$G \circlearrowright X \times E \qquad g\cdot(x,e) = (g\cdot x, g^{-1}\cdot x).$$

BG E/G is called the classifying space of G;

Equivariant cohomology is the cohomology of fibre spaces over BG

- existence of E ✓
- independence of the choice of E ✓

(deRham model...)

Groups of interest : linear algebraic groups.

$$G \hookrightarrow GL_n.$$

It suffices to find $\underset{\overset{\shortparallel}{E_n}}{E}$ for GL_n.

$V = \mathbb{C}^\infty$ = sequences of complex numbers stabilizing at 0.

$E = Fr(n,V)$ = n-tuples of linearly independant vectors in V.

$\circlearrowleft$

GL_n, free : ✓

E contractible ?

$$\begin{tikzcd} E \arrow[d] \\ Fr(n-1, V) \end{tikzcd}$$

fibration with fiber $Fr(1, \mathbb{C}^\infty) \times \mathbb{C}^{n-1}$.

By induction, it suffices to prove that $Fr(1, \mathbb{C}^\infty)$ is contractible.

$$\underbrace{Fr(1, \mathbb{C}^\infty)}^{= \mathbb{C}^\infty \setminus \{0\}} \times [0,1] \longrightarrow Fr(1, \mathbb{C}^\infty)$$

$$(x_i)_{i \in \mathbb{N}} \longmapsto (t, tx_0, tx_1, \dots)$$

Examples: • T - torus $\simeq (\mathbb{C}^*)^n$

$X^*(T) = \{T \to \mathbb{C}^*_{hom}\}$ characters of T.

abelian group.

$E = (\mathbb{C}^\infty \setminus \{0\})^n$

$H^*_T(pt) \simeq Sym^*(X^*(T))$

$H^*_T(pt) \simeq H^*_{\mathbb{C}^*}(pt)^{\otimes n}$ Künneth

$\simeq \mathbb{C}[x]^{\otimes n}$

$= \mathbb{C}[x_1, \dots, x_n]$

Chern classes of the tautological (line) bundles on $(\mathbb{P}^\infty)^n$

- GL_n

$$\begin{tikzcd} V \arrow[d, "\text{tautological vector bundle}"] \\ Gr(n,\infty) \end{tikzcd}$$

$$Fr(n,\infty)/GL_n \simeq Gr(n,\infty)$$

$$H^*_{GL_n}(pt) \simeq \mathbb{C}[\underbrace{c_1, \dots, c_n}_{\text{Chern classes of } V.}] \quad (\text{actually, true over } \mathbb{Z})$$

[Consequence of the computation of the cohomology of partial flag varieties G/P, G reductive and P parabolic subgroup].

- Projective space:

$$GL_n \simeq G = GL(V) \curvearrowright V \simeq \mathbb{C}^n$$

$$\begin{tikzcd} H^*_G(\mathbb{P}(V)) = H^*\left(\mathbb{P}(V)\times^G Fr(n,\infty)\right) \\ \mathbb{P}\left(V \times^G Fr(n,\infty)\right) \arrow[u, phantom, sloped, "\simeq"] \arrow[d, "\text{proj. bundle}"] \\ BGL_n \end{tikzcd}$$

$$\overset{\text{[prerequisite coh. fibre spaces]}}{=} H^*_G(pt)[\xi] \Big/ \left(\xi^n + c_1\xi^{n-1} + \dots + c_n\right).$$

ξ hyperplane class

$$c_i = c_i\left(V \times^G Fr(n,\infty)\right) =: c_i^G(V)$$

- $T = \overset{\text{diagonal}}{\text{max}}$ torus $\subset GL_n$

$$H^*_T(\mathbb{P}(V)) = H^*(\mathbb{P}(V)\times^T(\mathbb{C}^\infty\setminus\{0\})^n)$$

$$\begin{tikzcd} & \mathbb{P}(V\times^T(\mathbb{C}^\infty\setminus\{0\})^n) \arrow[u, phantom, "\wr"] \arrow[dl, squiggly, dash] \arrow[d, "\text{projective bdle}"] \\ V_1\oplus\cdots\oplus V_n & BT \end{tikzcd}$$

$$\simeq H^*(BT)[\xi] \Big/ \prod_{i=1}^n(\xi+t_i)$$

$$c(V_1\oplus\cdots\oplus V_n) = \prod c(V_i) \qquad c(V_i) = 1+t_i \in H^*(BT) = \mathbb{C}[t_1,\dots,t_n]$$

(Whitney formula)

② Localization theorem I

Thm: X smooth T-variety, T torus, $\# X^T < \infty$.

$$c = \prod_{p \in X^T} c_d^T(T_p X) \in H_T^*(pt).$$

$$\iota : X^T \hookrightarrow X$$

- $$\begin{tikzcd} H_T^*(X) \arrow[r, "\iota^*"] & H_T^*(X^T) \arrow[l, bend left, "\iota_*"] \end{tikzcd}$$
  ι^* is injective
- If $H_T^*(X)$ contains $\leq \# X^T$ classes restricting to a basis of $H^*(X)$ [practical criterion]

ι^* becomes iso after inverting c.

ι_*

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by restricting to a fiber.

$$\begin{tikzcd} H_T^*(X) = H^*(X \times^T E) \arrow[r] \arrow[d, "\leftarrow \text{fiber } X"] & H^*(X) \\ BT \end{tikzcd}$$

Localization theorem II

refinement : for all algebraic varieties

More notations :

$$\begin{array}{ccc} T & & X^*(T) =: M \\ \cup & & \cup \\ T(L) = \ker(L). & & L \text{ subgroup} \end{array}$$

$$\Big(= \bigcap_{\chi \in L} \ker \chi \Big)$$

$T(L)$ can be non-connected.

R coefficient ring

algebra of functions.

$$S(L) \subset H^*_T(pt) \simeq \overset{R}{\cancel{\mathbb{C}}}[\mathfrak{t}] \simeq \mathrm{Sym}(X^*(T)) \qquad X^*(T) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathfrak{t}^*$$

multiplicative set generated by $M \setminus L$.

General fact: X smooth $\Rightarrow$ X^A smooth for any torus A acting on X

Thm : X algebraic variety with torus action T.

$$\iota^* : H^*_T(X) \longrightarrow H^*_T(X^{T(L)})$$

is an isomorphism after inverting $S(L) \subset H^*_T(pt)$

$L=0$: $H^*_T(X) \xrightarrow{\iota^*} H^*_T(X^T)$ is an iso over the generic point of Spec $H^*_T(pt)$.

i.e. Ker ι^*, coker ι^* are torsion over $H^*_T(pt)$.

If $H^*_T(X)$ is free over $H^*_T(pt)$ ← integral domain.

$$\begin{tikzcd}
H^*_T(X) \arrow[r, "\iota^*"] \arrow[d, hook] & H^*_T(X^T) \arrow[d] \\
S^{-1}H^*_T(X) \arrow[r, "\sim", "S^{-1}\iota^*"'] & S^{-1}H^*_T(X^T)
\end{tikzcd}$$

③ Proof of the localization theorem

Step 1: $X^{T(L)} = \emptyset$

① $X = V \setminus V^{T(L)}$, V T-affine space linear action.

② X affine variety

③ X general

Step 2: X general w/ $X^{T(L)}$ possibly nonempty.

Long exact sequence of a pair (of $H^*_T(pt)$-modules)

$$\cdots \to H^i_T(X, X^{T(L)}) \to H^i_T(X) \xrightarrow{\iota^*} H^i_T(X^{T(L)}) \to \cdots$$

$\rightsquigarrow$ prove $S(L)^{-1} H^*_T(X, X^{T(L)}) = 0.$

Use approximation spaces:

$\exists$ principal T-bundle $E \to B$ s.t.

$$H^k_T(X, X^{T(L)}) = H^k(E \times^T X, E \times^T X^{T(L)})$$

$\cap$

$\mathcal{U}$ open ngh.

$\exists\, c \in H^*B$, c annihilates $H^k(E\times^T X \setminus E \times X^{T(L)})$.

$$\begin{tikzcd} H^k(E\times^T X \setminus E \times X^{T(L)}) \arrow[d, "\text{alg map}"] \\ H^k(\mathcal{U} \setminus E \times^T X^{T(L)}) \end{tikzcd}$$

$\Rightarrow c$ annihilates $\longrightarrow H^k(\mathcal{U} \setminus E \times^T X^{T(L)})$

$$H^k_T(X, X^{T(L)}) = H^k(E\times^T X,\ E\times^T X^{T(L)})$$
$$= \varinjlim H^k(E\times^T X, \mathcal{U})$$
$$\underset{\text{excision}}{=} \varinjlim_{n} H^k\big((E\times^T X)\setminus E\times^T X^{T(L)},\ \mathcal{U}\setminus E\times^T X^{T(L)}\big)$$

annihilated by c.

Proof of step 1:

① $X = V \setminus V^{T(L)}$

$$V = \bigoplus_{\chi\in M} V_\chi \qquad \text{wght space dec.}$$

$$= \underbrace{\bigoplus_{\chi\in L} V_\chi}_{\overset{\shortparallel}{V^{T(L)}}} \oplus \bigoplus_{\chi\notin L} V_\chi$$

$$\begin{tikzcd} X \arrow[d, "\text{projection}"] \\ V^{T(L)} \end{tikzcd} \quad = \quad \text{complement of zero section of} \quad \begin{tikzcd} V \arrow[d] \\ V^{T(L)} \end{tikzcd}$$

Gysin ex. seq $\Rightarrow$ $H^*_T(X) = 1/(c)$, $c = \prod_{\chi\notin L} \chi^{\dim V_\chi}$

T-equiv bundle

Eulerclass of normal $V^{T(L)}$ in V.

$c \in S(L) \Rightarrow S(L)^{-1} H_T^*(X) = 0$ ✓

② $\exists$ T-equiv $X \underset{\circlearrowleft\, T}{\hookrightarrow} \underset{\text{lin.}}{V}$

$$X \hookrightarrow V \setminus V^{T(L)}$$

$\exists c \in S(L)$, $c \cdot H_T^*(V \setminus V^{T(L)}) = 0$

$\Rightarrow c \cdot H_T^*(X) = 0.$

③ $X = \bigcup_{i=1}^{N} U_i$ covering by affine, T-inv opens.

$\exists c_i$ annihilating $H_T^*(U_i)$

Mayer-Vietoris $\Rightarrow \prod_{i=1}^{N} c_i$ annihilates $H_T^*(X)$.

∎

Image of the restriction map

$$\iota^* : H_T^*(X) \longrightarrow H_T^*(X^T)$$

coefficient ring R UFD

Two characters of T are relatively prime if

- non-parallel
- their coefficients are relatively prime

irreducible factor : $f \in \Lambda$ image of prime in $\mathbb{Z}$
via $\mathbb{Z} \longrightarrow \Lambda = H_T^*(pt)$
or of a primitive character
under $X^*(T) \longrightarrow H_T^*(pt)$.

f is factor $\longrightarrow$ $L_f \subset M$ subgroup of characters divisible by f.

$T(f) \subseteq T$ subtorus.

$f = \chi$ primitive $\Rightarrow$ $T(f)$ subtorus of codim one
$f = p$ prime in $\mathbb{Z}$ $\Rightarrow$ $T(f)$ = elements of order p

Thm (Chang–Skjelbred)

$T \circlearrowleft X$

$H^*(X^T)$ free R-module

$H^*_T(X)$ free Λ-module $= \bigoplus_{i=1}^{r} \Lambda e_i$

$$\operatorname{im}\left(\iota^* : H^*_T(X) \to H^*_T(X^T)\right)$$

$$= \bigcap_f \left(\operatorname{im} H^*_T(X^{T(f)}) \to H^*_T(X^T)\right)$$

$\forall$ irr factor f.

Proof: general localization thm.

$\subset$ is trivial via $X^T \subset X^{T(f)} \subset X$.

$$\supset \quad \underset{\overset{?}{H_T(X^T)}}{\alpha} \ \underset{\text{Eq}\nearrow}{\in} \ \bigcap_f \operatorname{im}(\qquad).$$

$H^*_T(X) = H^*_T(X^T)$ after inverting $H^*_T(pt) \setminus \{0\}$.

$\Rightarrow \exists g \in \underbrace{H^*_T(pt)}_{} \setminus \{0\}, \quad g\alpha \in H^*_T(X)$ $(*)$

UFD → take g s.t. $(*)$ does not hold for proper factors of g.

$$g\alpha = \sum_{i=1}^{r} a_i e_i$$

If $\alpha \notin \mathrm{im}(*)$, g not a unit.

f irr factor of g

f, a_1 coprime (WLOG)

<u>general loc thm</u> $S(f)^{-1} H_T^*(X) = S(f)^{-1} H_T^*(X^{T(f)})$

$S(f)$ = characters not div by f.

$\exists\, \psi_f \in S(f)$, $\psi_f \alpha \in H_T^*(X)$.

$\psi_f \alpha = \sum_{i=1}^{r} b_i e_i$; f is not div by f

$g\psi_f\alpha$, has coef on e_1 $\underbrace{g b_1}_{\text{div by } f} = \underbrace{\psi_f a_1}_{\text{not div by } f}$.

∎

Corollary (GKM)

X nonsing

X^T finite

$H_T^*(X)$ free over Λ

$\forall p \in X^T$, the weights on T_pX are relatively prime

non-parallel and coefficients relatively prime in coeff ring.

Then $(u_p)_{p\in X^T} \in H_T^*(X^T)$ lies in the image of ι^* iff for each T-curve $C_{pq} \simeq \mathbb{P}^1$ connecting $p, q \in X^T$, $u_p - u_q$ is divisible by $\pm \chi_{pq}$, character of C_{pq}

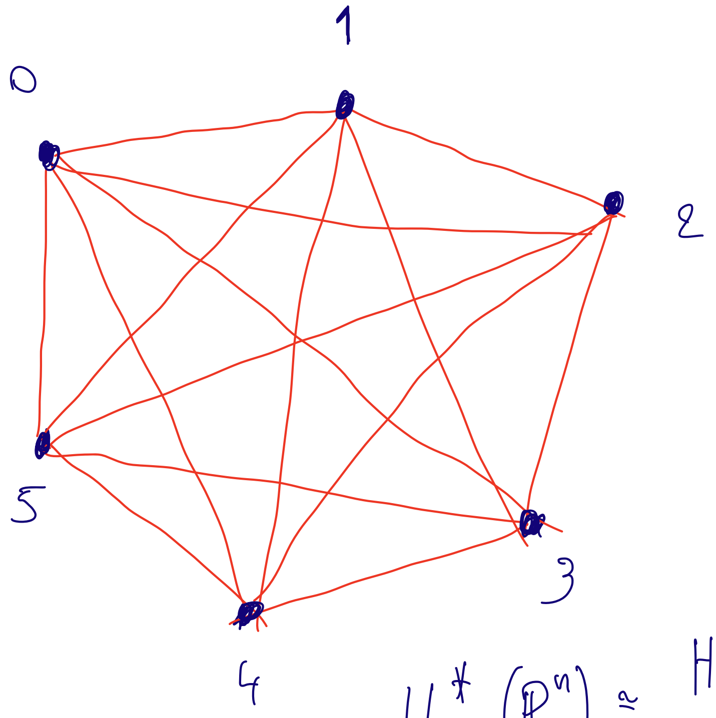
GKM-variety : finitely many fixed points
finitely many T-curves

closure of a 1-dim T-orbit in X

↳ $Tx \overset{\text{isomorphism}}{\simeq} \mathbb{C}^*$
gives a character of T, $\pm\chi$.

eg: $\mathbb{P}^n \circlearrowleft T \simeq \mathbb{C}^{n+1}$ n+1 fixed points

$t_0 \dots t_n$



arrow between i and j ⤳ character $t_i - t_j$

$n = 5$

$$H_T^*(\mathbb{P}^n) \simeq H_T^*(pt)[\xi] \Big/ \prod_{i=0}^{n}(\xi + t_i)$$

$$\begin{tikzcd} H_T^*(\mathbb{P}^n) \arrow[r, "\iota^*"] & \bigoplus_{i=0}^{n} H_T^*(p_i) \\ & \mathbb{C}[t_0, \dots, t_n] \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

$$\sum_{j=0}^{n-1} P(\underline{t})\,\xi^j \longmapsto \left(\sum_{j=0}^{n-1} P(\underline{t})\,(-t_i)^j \right)_i$$

$$\iota_{p_i}^* \xi = -t_i$$

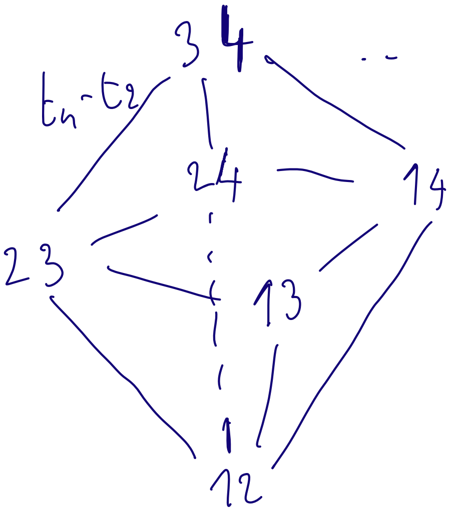
$$c_1^T\left(\mathcal{O}(-1)\big|_{p_i}\right) = t_i$$

- image ⊂ divisibility condition is clear
- (boring) exercise to check that this is indeed the image.

eg: $Gr(2,4) \circlearrowleft \underset{\shortparallel\, T}{(\mathbb{C}^*)^4}$ rescale coordinate axes

$V \subset \mathbb{C}^4$ stable: $\binom{4}{2}$ choices

$$= \frac{4\cdot 3}{2} = 6.$$



... diff of the one changing.

tautologique on $Gr(2,4)$

$$0 \longrightarrow S \to \underline{\mathbb{C}^4} \longrightarrow Q \to 0$$

$$H_T^*(Gr(2,4)) \simeq H_T^*(pt)\big[\underset{\underset{c_1}{\shortparallel}}{c_1(S)}, \underset{\underset{c_2}{\shortparallel}}{c_2(S)}, \underset{\underset{\tilde{c}_1}{\shortparallel}}{c_1(Q)}, \underset{\underset{\tilde{c}_2}{\shortparallel}}{c_2(Q)}\big]$$

$$c(S)\cdot c(Q) = \underset{\underset{\prod_{i=1}^{4}(1+t_i)}{\shortparallel}}{c^T(V)}$$

$\underset{\shortparallel}{(i,j,k,l)}$

$\{i,j\} \subset \{1,\dots,4\}$ 2 elts set.

$$c(i_{ij}^* S) = (1+t_i)(1+t_j)$$

$$c(i_{ij}^* Q) = (1+t_k)(1+t_l)$$

$$H_T^*(Gr(2,4)) \longrightarrow \bigoplus_{\{i\neq j\}\subset\{1,\dots,4\}} H_T^*(p_{ij})$$

$$\begin{cases} C_1 \longmapsto t_i + t_j \\ C_2 \longmapsto t_i t_j \\ \tilde{C}_1 \longmapsto t_k + t_l \\ \tilde{C}_2 \longmapsto t_k t_l . \end{cases}$$

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ex of image when the characters are not relatively prime

$T \circlearrowright \mathbb{P}^2$ characters $(0, x, 2x)$; x primitive nonzero.

$$H_T^*(\mathbb{P}^2) = H_T^*(pt)[\xi] \big/ \xi(\xi+x)(\xi+2x) \xrightarrow{\iota^*} H_T^*(pt)^{\oplus 3}$$

$$\xi \longmapsto (0, -x, -2x)$$

image is the subring of triples (u_1, u_2, u_3) such that

(1) $u_2 - u_1$; $u_3 - u_2$ div by x
$u_3 - u_1$ by $2x$] factor through the curve between p_i and p_j.

(2) $u_1 - 2u_2 + u_3$ by $2x^2$.

↳ from the integration formula

$$p_*(u) = \frac{u_1}{2x^2} + \frac{u_2}{-x^2} + \frac{u_3}{2x^2}$$

(see next paragraph)

$$= \frac{u_1 - 2u_2 + u_3}{2x^2} \in H_T^*(pt) \quad \Rightarrow 2x^2 / u_1 - 2u_2 + u_3 .$$

Proves necessity of the conditions.

Sufficiency: exercise.

④ Equivariant formality.

General criteria garanteeing the hypotheses of the localization thm I.

Def $X \circlearrowleft G$ k-formal if $H^*_G(X) \simeq H^*(X) \otimes_k H^*_G(pt)$

$\underline{k}$ coefficients ring.

formality is implied by the criterions:

- $H^i(X)$ fin dim $\forall i$
- $H^i_G(X) \longrightarrow H^i(X)$ surjective (k field).

Eq. formality is an other name for the degenerescence of the Serre spectral sequence of the fibration $X \to X \times^G E \to BG$.

Thm: $X \circlearrowleft T$, $\# X^T < \infty$, X smooth and projective.

Then X is eq. formal (over $\mathbb{Z}$)

Proof: BB decomposition

Thm 14.1, [GKM]: $\exists$ sufficient conditions for equivariant formality.

- $H^*(X, \mathbb{Q})$ is even; G connected linear algebraic group
- X nonsing proj, T torus (Bialinicky-Birula!)

* X projective alg. var + $H^*(X;\mathbb{Q})$ pure -

ex of a non-equivariantly formal space.

$X = \mathbb{P}^1 / 0 \sim \infty$. $\mathbb{C}^* = T \circlearrowleft \mathbb{P}^1$ $z \cdot [a, b] = [a : zb]$

$$H_T^*(X) = \underset{\substack{\| \\ \mathbb{Z}[t]}}{H_T^*(pt)}[\alpha] \Big/ (\alpha^2, t\alpha)$$

See that $H_T^*(X) \big/ \text{torsion} = H_T^*(pt)$

$$H^*(X) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$$

$H_T^*(X) \longrightarrow H^*(X)$ is not surjective (not surjective in degree 2!)

Restriction map: $H_T^*(X) \longrightarrow H_T^*(X^T) = H_T^*(pt)$

$$\alpha \longmapsto 0$$

$$\begin{tikzcd}
\{0, \infty\} \arrow[r] \arrow[d] & pt \arrow[d] \\
\mathbb{P}^1 \arrow[r] & X
\end{tikzcd}$$

$\alpha \in H^1_T(X)$! and degree considerations.

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⑤ Integration formula : T gives symmetries of X.

→ exploit this symmetries to simplify / make possible some computations

$S \subseteq \Lambda$ s.t.

$$H_T^*(X^T) \xrightarrow{\iota_*} H_T^*(X) \xrightarrow{\iota^*} H_T^*(X^T)$$

are iso after inverting S.

$f : X \longrightarrow Y$ T- equivariant

$$\begin{tikzcd} X^T \arrow[r] \arrow[d, phantom, "\shortparallel"] & Y^T \arrow[d, phantom, "\shortparallel"] \\ \bigsqcup P & \bigsqcup Q \end{tikzcd}$$

connected components

$P \xrightarrow{f_P} Q$ induced by f.

$$\begin{tikzcd} P \arrow[r, hook, "i_P"] \arrow[d, "f_P"'] & X \arrow[d, "f"] \\ Q \arrow[r, hook, "i_Q"'] & Y \end{tikzcd}$$

(*)

Thm (integration formula)

For any $u \in H^*_T(X)$, $Q \subset Y^T$,

$$f_*(u)|_Q = \boxed{c^T_{top}(N_{Q/Y}) \sum_{\substack{p \in \pi_0(X^T) \\ f(p) \subset Q}}' (f_p)_* \left(\frac{u|_p}{c^T_{top}(N_{p/X})} \right)}.$$

Proof:

Assume $u = (\iota_p)_*(z)$ $\quad z \in H^*_T(p)$ $\quad p \in \pi_0(X^T)$

🟥 $= \iota^*_Q \underbrace{f_*(\iota_p)_*}_{(\iota_Q)_*(f_p)_*} (z)$ $\quad$ diag $(*)$

$\underbrace{\iota^*_Q (\iota_Q)_*}_{c^T_{top}(N_{Q/Y})}$ self-int. formula

$= c^T_{top}(N_{Q/Y}) (f_p)_*(z)$ $\quad (p \subset Q)$

🟢 $= \iota_p^*(\iota_p)_*(z) = c^T_{top}(N_{p/X}) \cdot z$

$u_{p'} = 0$ if $p \neq p'$

So $$\Box = c^T_{top}(N_{Q/Y})\,(f_P)_*\left(\frac{\cancel{c^T_{top}(N_{P/X})}\cdot \gamma}{\cancel{c^T_{top}(N_{P/X})}}\right)$$

$\blacksquare$

Write the integration formula when

- $Y = pt$
- X, Y have finitely many fixed points

Remark: • If $f : X \to Y$ is not proper but $f^T : X^T \to Y^T$ is proper, we can define the integration by the r.h.s of Atiyah-Bott formula, although the lhs is not well-defined.

- If $f : X \to pt$ proper, $f_* : H^*_G(X) \to H^*_G(pt)$ so integrals don't have poles: denominators in Atiyah-Bott formula compensate each other.

- If X is possibly non-compact but $\alpha \in H^*_G(X)$ is a compact class, that is $\alpha = \iota_* \beta$ for $\iota_* : Y \hookrightarrow X$

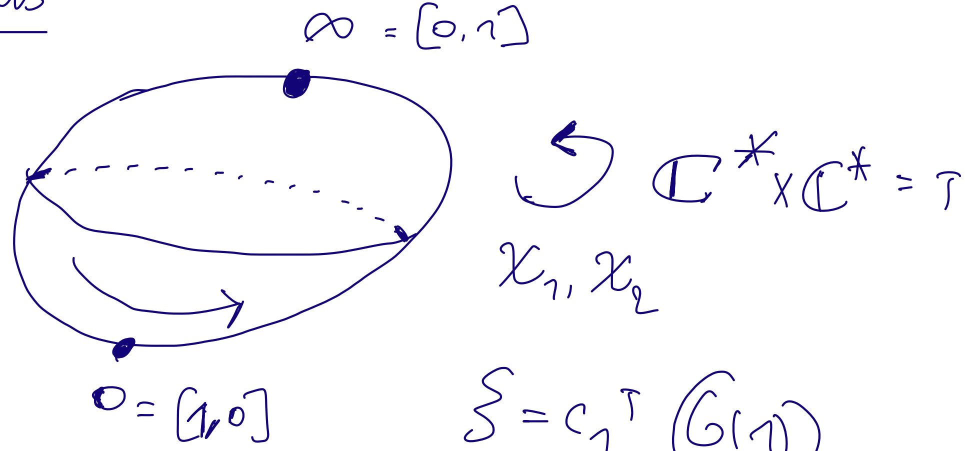
inclusion of a closed, proper $\underbrace{\text{subvariety}}_{\text{T-invariant}}$,

$f: X \to pt$;

$f_*\alpha$ defined by the rhs of Atiyah-Bott formula doesn't have poles.

⑥ Computations of integrals

Ex 1 $\mathbb{P}^1 =$



$\xi = c_1^T(\mathcal{O}(1))$

$$H^*_T(\mathbb{P}^1) = H^*_T(pt)[\xi] \big/ (\xi + x_1)(\xi + x_2)$$

$p : \mathbb{P}^1 \longrightarrow pt$

$p^T : \{0, \infty\} \longrightarrow pt$.

$\xi \in H^*_T(\mathbb{P}^1)$

$$p_*(\xi) = \frac{i_0^*\xi}{\underbrace{c^T_{top}(T_0\mathbb{P}^1)}_{x_2 - x_1}} + \frac{i_\infty^*\xi}{\underbrace{c^T_{top}(T_\infty\mathbb{P}^1)}_{x_1 - x_2}}$$

$$i_0^*\xi = -x_1$$

$$i_\infty^*\xi = -x_2$$

$$= \frac{-x_1}{x_2 - x_1} + \frac{-x_2}{x_1 - x_2}$$

$$= 1 .$$

In fact, $\xi = [0]^T = [\infty]^T$ hyperplane class

so $\varphi_*(\xi) = (\varphi \circ i_0)_* (1) = 1$!