Perverse sheaves with nilpotent singular support for curves and quivers

Italian Representation Theory Seminar

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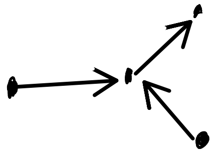
Lucien Hennecart

$Q = (I, \Omega)$ quiver



$\mathrm{Rep}_Q(\mathbb{C})$ – category of reps of Q over $\mathbb{C}$

$\mathcal{M}_Q = \bigsqcup_{d \in \mathbb{N}^I} \mathcal{M}_{Q,d}$ stack of objects

Π_Q = preprojective algebra of Q

$\mathrm{Rep}_{\Pi_Q}(\mathbb{C})$ – category of reps of Π_Q over $\mathbb{C}$

$\mathcal{M}_{\Pi_Q} \simeq T^*\mathcal{M}_Q$ stack of objects

X smooth projective curve / $\mathbb{C}$.

$\mathrm{Coh}(X)$ – category of coherent sheaves on X

$\mathcal{C}oh(X)$ stack of objects

$\mathrm{Higgs}(X)$ = category of Higgs sheaves on X

$\mathcal{H}iggs(X) \simeq T^*\mathcal{C}oh(X)$ stack of Higgs sheaves

$\mathcal{Q} \subset D^b_c(\mathcal{M}_Q)$

= a certain category of constructible complexes on $\mathcal{M}_Q$

"Lusztig complexes"

$\Lambda \subset T^*\mathcal{M}_Q$ - a certain closed, conical, Lagrangian substack

"Lusztig nilpotent stack"

$CC : K_0(\mathcal{Q}) \longrightarrow \mathbb{Z}[\mathrm{Irr}\,\Lambda]$

the characteristic cycle map.

$\mathcal{Q} \subset D^b_c(\mathcal{C}oh(X))$

"spherical Eisenstein complexes"

$\Lambda \subset \mathcal{H}iggs(X)$ the global nilpotent cone

$CC : K_0(\mathcal{Q}) \longrightarrow \mathbb{Z}[\mathrm{Irr}\,\Lambda]$

Questions :

① What are the properties of CC ?

② If $D^b_c(\mathcal{M}, \Lambda)$ $[\mathcal{M} = \mathcal{M}_Q \text{ or } \mathrm{Coh}(X)]$

$$\begin{array}{c} D^b_c(\mathcal{M}, \Lambda) \\ \cap \\ D^b_c(\mathcal{M}) \end{array}$$

category of complexes with singular support $\subset \Lambda$,
can we compare $K_0(Q)$ & $K_0(D^b_c(\mathcal{M}, \Lambda))$?

$\subset$

<u>always</u>

In both cases,

$K_0(\mathcal{Q})$ is endowed with a multiplication and a comultiplication : induction & restriction of complexes.

$\longrightarrow$ bialgebra

$\mathbb{Z}[\mathrm{Irr}\,\Lambda]$ also has a multiplication and comultiplication coming from the cohomological Hall algebra.

General properties of CC

Vasserot

- CC is an algebra map (not written yet to my knowledge)

- CC is surjective (Sala - Schiffmann)

- CC is injective for quivers & curves of genus $0 \leq g \leq 1$

[conjecturally for all curves]. (genus 1 : H)

- CC is a bialgebra map for quivers. (Davison)

[conjecturally, also for curves].

- For quivers, $K_0(Q) \simeq \mathbb{Z}[\mathrm{Irr}\,\Lambda] \simeq U_{\mathbb{Z}}(\mathfrak{n}_+)$

divided power $\mathbb{Z}$-form of unipotent enveloping algebra.

Quivers

$$Q = (\underbrace{I}_{\text{vertices}}, \underbrace{\Omega}_{\text{arrows}})$$

$$\begin{tikzcd} i \arrow[r, "\alpha"] & j \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\gamma"'] & k \end{tikzcd}$$

$d \in \mathbb{N}^I$: $\mathcal{M}_{Q,d} = E_d / G_d$ stack quotient.

$$E_d = \bigoplus_{\alpha : i \to j \in \Omega} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

$$G_d = \prod_{i \in I} GL_{d_i}$$

Perverse sheaves on $\mathcal{M}_{Q,d}$

$d \in \mathbb{N}^I \quad ; \underline{d} = (d_1, d_2, \dots, d_s) \in (\mathbb{N}^I)^s \quad$ s.t. $\quad \sum d_i = d$.

$V = \mathbb{C}^d \quad$ $\mathbb{N}^I$-graded vector space of dimension d.

$\tilde{\mathcal{F}}_{\underline{d}}$ = stack of pairs $(F_\bullet, x)$ $\left\{ \begin{array}{l} 0 = F_0 \subset \dots \subset F_s = F \\ \dim F_i / F_{i-1} = d_i \\ x F_i \subset F_i \end{array} \right.$

Smooth stack

$$\begin{array}{rccc} \pi_{\underline{d}} : & \tilde{\mathcal{F}}_{\underline{d}} & \longrightarrow & \mathcal{M}_{Q,d} \\ & (F_\bullet, x) & \longmapsto & x \end{array}$$

projective.

$\Longrightarrow \quad (\pi_{\underline{d}})_* \underline{\mathbb{C}}$ is a semisimple complex on $\mathcal{M}_{Q,d}$.

i.e. $(\pi_{\underline{d}})_* \underline{\mathbb{C}} = \bigoplus_{d \in \mathbb{Z}} \mathcal{F}[d]$ for semisimple perverse sheaves $\mathcal{F}$ on $\mathcal{M}_{Q,\underline{d}}$.

$\subset D^b_c(\mathcal{M}_{Q,d})$

$\mathcal{Q}_{\underline{d}}$ = full triangulated category of semisimples complexes on $\mathcal{M}_{Q,\underline{d}}$ generated by the simple constituents of $(\pi_{\underline{d}})_* \underline{\mathbb{C}}$. for various $\underline{d}$.

$\mathcal{P}_{\underline{d}}$ subcategory of perverse sheaves

Description of $\mathcal{P}_d$ for finite type and affine quivers

Q finite type quiver

$d \in \mathbb{N}^I$. E_d / G_d (set-theoretic quotient) is finite :

$$E_d = \bigsqcup_{\substack{G \subset E_d \\ G_d\text{-orbit}}} G$$

Each orbit is equivariantly simply connected :

no nontrivial G_d-equivariant local systems on G

$\updownarrow$ 1 : 1

(group of connected components of $\mathrm{Aut}(x)$ for $x \in G$.)

$\Rightarrow$ The only simple equivariant perverse sheaves on $\mathcal{M}_{Q,d}$ are the $IC(\mathbb{O})$ for $\mathbb{O} \subset E_d$ orbit.

$\ast$ For $\mathbb{O} \subset E_d$, can find $\underline{d} \in (\mathbb{N}^I)^s$ s.t.

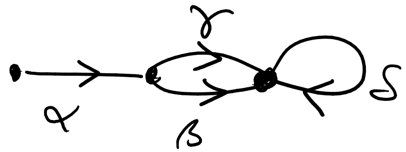
$\pi_{\underline{d}}$ is a resolution of $\overline{\mathbb{O}}/G_d$.

$$\Rightarrow IC(\mathbb{O}/G_d) \overset{\oplus}{\subset} (\pi_{\underline{d}})_* \underline{\mathbb{C}}[\dim \mathbb{O}/G_d].$$

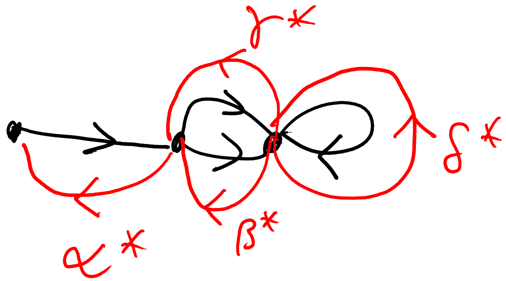
So $\mathcal{P}_d = \langle IC(\mathbb{O}/G_d),\ \mathbb{O} \subset E_d \text{ a } G_d\text{-orbit} \rangle$

The nilpotent stack

$Q = (I, \Omega)$ quiver



$\overline{Q} = (I, \Omega \sqcup \Omega^*)$ doubled quiver



$$\Pi_Q = \mathbb{C}\overline{Q} \Big/ \sum_{\alpha \in \Omega} [\alpha, \alpha^*]$$ preprojective algebra

$d \in \mathbb{N}^I$

$\mathcal{M}_{Q,d}$ = stack of d-dim reps of Π_Q

$\supset \Lambda_d$ stack of d-dim nilpotent reps of Π_Q.

the action of a sufficiently long path in $\overline{Q}$ is trivial.

The nilpotent stack of finite type quivers

$Q = (I, \Omega)$ finite type quiver

$$\Lambda_d = \bigcup_{\substack{G \subset E_d \\ G_d\text{-orbit}}} T^*_G E_d / G_d \subset \mathcal{M}_{\Pi_Q, d}.$$

$=$ equality

$$\mathrm{Irr}(\Lambda_d) = \left\{ \overline{T^*_G E_d} / G_d \; : \; G \;\; G_d\text{-orbit} \right\}$$

$\ulcorner$ $\mathcal{M}_{\Pi_Q, d}$ can be constructed by stacky symplectic reduction

$$\mu_d : T^* E_d \longrightarrow (\mathfrak{gl}_d)^* \underset{\text{trace}}{\simeq} \mathfrak{gl}_d$$

$$(x_\alpha, x_\alpha^*)_{\alpha \in \Omega} \longmapsto \sum_{\alpha \in \Omega} [x_\alpha, x_{\alpha^*}]$$

$$\mathcal{M}_{\pi_Q, d} = \mu_d^{-1}(0) / G_d .$$

general fact $= \bigcup_{G \text{ } G_d\text{-orbit}} T^*_G E_d \Big/ G_d$.

Characteristic cycle and singular support of constructible complexes

· Many possible definitions (Kashiwara-Schapira, ...)

Axiomatic definition

The characteristic cycle map

X smooth $\mathbb{C}$-variety

$D^b_c(X)$ category of constructible complexes on X

$\mathbb{Z}[\mathrm{Lagr}^{\mathbb{C}^*}(T^*\mathbb{C})]$ - functions $\mathrm{Lagr}^{\mathbb{C}^*}(T^*X) \longrightarrow \mathbb{Z}$ with finite support.

$\mathrm{Lagr}^{\mathbb{C}^*}$ $\|$ closed, conical, Lagrangian subvarieties of T^*X

$$CC : K_0(D^b_c(X)) \longrightarrow \mathbb{Z}[\mathrm{Lagr}^{\mathbb{C}^*}(T^*X)]$$

- morphism of abelian groups
- functoriality w.r.t. smooth pull-backs and proper push-forwards.
- normalization: $CC(\mathcal{L}) = [T^*_X X]$

$\mathcal{L}$ local system on X.

Singular support (of a perverse sheaf)

$\mathcal{F} \in Perv(X)$.

$$SS(\mathcal{F}) = supp(CC(\mathcal{F})) \subset T^*X$$

Closed, conical, Lagrangian subvariety.

$p : T^*X \longrightarrow X$ cotangent bundle.

$\mathcal{F} \in \mathrm{Perv}(X)$

Useful properties : • $p(SS(\mathcal{F})) = \mathrm{supp}\,\mathcal{F} \subset X$

• $SS(\mathcal{F}) = T^*_X X \implies \mathcal{F} = \mathcal{L}[\dim X]$

$\mathcal{L}$ local system on X.

• functoriality smooth pull-back

proper pushforward.

Theorem (Lusztig) If Q is a finite type quiver,

$$K_0(Q) = K_0\left(D^b_c\left(\mathcal{M}_{Q,d}\,;\,\Lambda_d\right)\right)$$

Proof: $\mathcal{F}$ simple G_d-equiv. perverse sheaf on E_d; $SS(\mathcal{F}) \subset \Lambda_d$

?

$\Rightarrow \mathcal{F} = IC(\mathcal{O})$ for some G_d-orbit $\mathcal{O} \subset E_d$.

But $SS(\mathcal{F}) \subset \Lambda_d \Rightarrow \exists\, \mathcal{O} \subset E_d$, $\operatorname{supp}\mathcal{F} = \overline{\mathcal{O}}$

$\Rightarrow$ (G_d-equivariance) $\mathcal{F}|_{\mathcal{O}}$ is a G_d-equiv loc sys

$\Rightarrow \mathcal{F} = IC(\mathcal{O})$. ∎

Theorem (H) If Q is an affine quiver

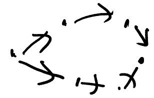
$$K_0(\mathcal{Q}) = K_0\left(D^b_c\left(\mathcal{M}_{Q,d}; \Lambda_d\right)\right)$$

Ingredients:
- explicit description of $\mathcal{P}_d$
- explicit description of Λ_d

} using the rep. theory of affine quivers

- cyclic quivers to describe an (analytic) neighbourhood of non-homogeneous tubes in $\mathcal{M}_{Q,d}$.
- consider a bigger category of perverse sheaves for cyclic quivers and a bigger nilpotent stack

Repr. theory of affine quivers

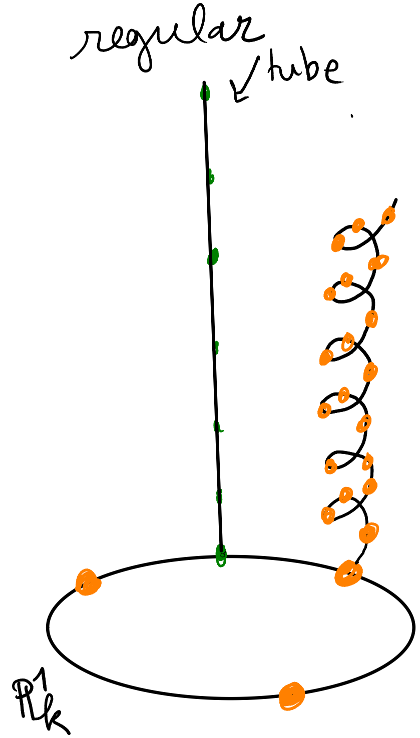
Q affine ADE quiver. A  D, E - -

k arbitrary field.

Auslander-Reiten quiver of Q

P_i ($i \in I$)
projective

preprojective
$\tau^{-N} P_i$



preinjective
$\tau^{N} I_i$

I_i, $i \in I$
injective

combinatorial part of the category: independent of k

Hom's $\xrightarrow{\quad}$ D

$\xleftarrow{\quad}$ Ext's

- closed point of $\mathbb{P}^1_k$
- special point of degree 1.

Regular part

$\delta \in \mathbb{N}^I$ indivisible imaginary root of Q.

$r \geqslant 1$

$$\begin{tikzcd}
\mathcal{M}_{Q,r\delta} \arrow[d, "f"'] & \supset & \mathcal{M}'_{Q,r\delta} \arrow[d] \arrow[ll, phantom, "\urcorner" description, shift right=6] & \simeq & \left(\mathfrak{gl}_r/GL_r\right)' \arrow[r, hook] \arrow[d] \arrow[r, phantom, "\ulcorner" description, shift right=6] & \mathfrak{gl}_r/GL_r \arrow[d] \\
S^r\mathbb{P}^1 & \supset & S^r(\mathbb{P}^1 \setminus D) & \simeq & S^r(\mathbb{A}^1 \setminus D') \arrow[r, hook] & S^r\mathbb{A}^1
\end{tikzcd}$$

(assume $|D| \geqslant 1$)

But this is not sufficient: if $d \in D$, need to describe a neighbourhood of $f^{-1}((d, \ldots, d))$ inside $\mathcal{M}_{Q, rs}$.

$\longrightarrow$ use cyclic quivers

Curves :

Theorem (H) $CC : K_0(\mathcal{Q}) \longrightarrow \mathbb{Z}[\mathrm{Irr}\,\Lambda]$ is injective for curves of genus $0 \leq g \leq 1$.

$g = 0$: analogous to finite type quivers.

the stack $\mathrm{Bun}_{(r,d)}$ locally finitely many $\mathbb{C}$-points

$X = \mathbb{P}^1$

$$\mathcal{P}_{(r,d)} = \left\langle IC\left(\mathcal{F} / \mathrm{Aut}(\mathcal{F}) \right) : \begin{array}{l} \mathcal{F} \text{ rk } r \\ \text{degree } d \\ \text{vector bdle on } \mathbb{P}^1 \end{array} \right\rangle$$

$$\Lambda_{r,d} = \bigcup_{\substack{[\mathcal{F}] = (r,d) \\ \text{vector bdle}}} T^*_{\mathcal{F}/\mathrm{Aut}(\mathcal{F})} \mathrm{Coh}_{(r,d)}(X)$$

$g=1$: X is an elliptic curve

- Prove the result for torsion sheaves ($\rightsquigarrow$ Springer theory for $\mathfrak{gl}_d$, $d \geq 0$)

- Harder-Narasimhan stratification

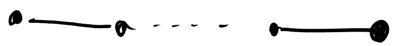
- $\mathrm{Coh}^{ss}_{\alpha}(X) \simeq \mathrm{Coh}_{(0,\gcd(\alpha))}$ $(\forall \alpha)$

- glue everything using functoriality of CC.

$g \geq 2$: seems difficult!

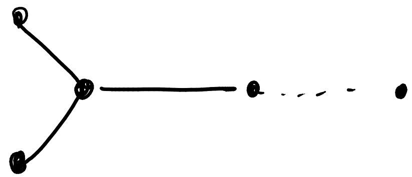
Finite type quivers

A_n

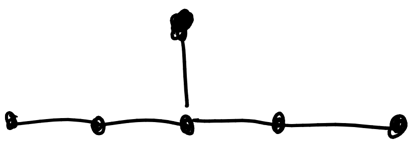


n vertices

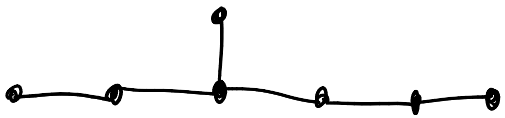
D_n



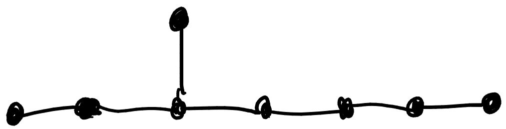
E_6



E_7

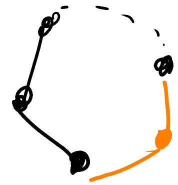


E_8



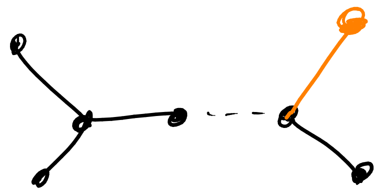
Affine quivers

$\tilde{A}_n$

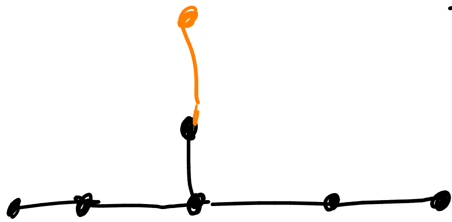


$n+1$ vertices

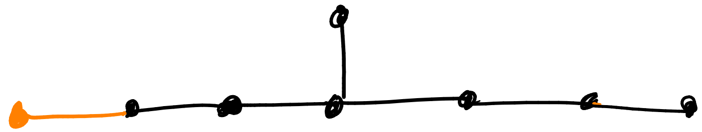
$\tilde{D}_n$



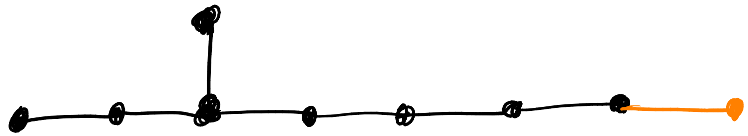
$\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$



Thank you for your

attention