<u>Hodge seminar</u> - 7th october 2021

The number of isoclasses of absolutely indecomposable representations of the modular group is a polynomial (jt work w/ Fabian Korthauer)

The title has been cooked to attract people and I think it did the job well. Things work in a bigger generality → arithmetic subgp of $SL_2(\mathbb{Z})$, In this talk, I'll present classical new ideas to prove a result which I think is interesting.

$G = PSL_2(\mathbb{Z})$ = modular group. running example

- representations of $PSL_2(\mathbb{Z})$ over $\mathbb{F}_q$, $2,3 \nmid q$ and $\mathbb{F}_q$ contains a primitive 3-rd root of unity ⚠ $\boxed{\text{Formula}}$

<u>Why?</u> : $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ w/ generators $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

$$PSL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \right\} \Big/ \{\pm 1\}$$

can be shown using the Ping-Pong lemma which is criterion for a product of groups to be free.

- Why are we interested in the rep th of $PSL_2(\mathbb{Z})$? **Why not.**
- Why counting representations of $PSL_2(\mathbb{Z})$? In this talk, we count all/indec/absindec reps of $PSL_2(\mathbb{Z})$ over $\mathbb{F}_q$. $\boxed{\text{F. Korthauer}}$ ← (also, possible to count ss/simples/abs simple)

The Motivation comes from Kac work in the 80's who counted the number of (isoclasses of) (absolutely) indecomposable representations of a quiver over a finite field.

$Q =$ quiver

$= (I, \Omega)$ — vertices I, arrows Ω

$$M_{Q,d}(q) = \#\left\{ \text{reps of } Q \text{ over } \mathbb{F}_q \text{ of dim } d \right\} \Big/ \sim$$

$$I_{Q,d}(q) = \#\left\{ \text{indec reps} \right\} \Big/ \sim$$

$$A_{Q,d}(q) = \#\left\{ \text{absindec reps} \right\} \Big/ \sim$$

+ analogous defn's for G

Kac: $M_{Q,d}$, $I_{Q,d}$, $A_{Q,d}$ $\in \mathbb{Q}[q]$

$\rotatebox{90}{$\subset$}$ $\rotatebox{90}{$\subset$}$

$\mathbb{Z}[q]$

conj: $A_{Q,d}(q) \in \mathbb{N}[q]$, thm: HLRV 2013.

Davison different method

Main ~~feature of Rep Q~~ : it is of homological dimension 1:

$Ext^j = 0 \quad j \geq 2$

stack $\underline{Rep\,Q} = \bigsqcup_{d \in \mathbb{N}^I} Ed / GLd$

product of GL groups

affine space

has rational fractio count and has has pure compactly supported cohomology.

- $PSL_2(\mathbb{Z})$ is a virtually free group: it has a free subgroup of finite order.

The kernel of the map $PSL(2,\mathbb{Z}) \to PSL(2, \mathbb{Z}/2\mathbb{Z})$ does the job.

$\| \; \Gamma_0(2)$, freely gen by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ & $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

(Karrass, Pietrowski, Solitar)

Thm: $\mathcal{G}$ group. ① $\mathcal{G}$ is virtually free $\Leftrightarrow$ $\mathcal{G} \simeq \pi_1$(finite graph of finite groups)

$\mathcal{G}$ has no K-torsion

② (Dicks) $K[\mathcal{G}]$ is hereditary $\Leftrightarrow$ $\mathcal{G} \simeq \pi_1$(graph of finite groups)

③ (le Bruyn) $\mathcal{G}$ f.g group, k field. $k[\mathcal{G}]$ is formally smooth $\Leftrightarrow$ $\mathcal{G}$ v.f and has no K-torsion.

A formally smooth algebra $\Rightarrow$ the rep. schemes of A are smooth

$Rep_n(A)$, $n \in \mathbb{N}$.

$Rep_n(A)$ rep. the functor comm. k-alg $\longrightarrow$ Sets

$B \mapsto \underset{k\text{-alg}}{Hom}(A, M_n(B))$

Representation stack

$GL_n \curvearrowright Rep_n(A)$
over k

$\mathcal{R}ep_n(A) := Rep_n(A)/GL_n$ rep. stack
is a smooth Artin stack.

Easy fact

$$\mathcal{R}ep_n H * K \simeq \mathcal{R}ep_n H \underset{pt/GL_n}{\times} \mathcal{R}ep_n K$$

but H, K are finite groups.

$\{M_1 \ldots M_h\}$ representations of H $/\sim$ of dim n

$\{N_1 \ldots N_k\}$ reps of K $/\sim$ of dim n

$$\mathcal{R}ep_n H = \coprod_{i=1}^{h} pt/\underbrace{Stab\, M_i}_{L_i}$$

$$\mathcal{R}ep_n K = \coprod_{i=1}^{k} pt/\underbrace{Stab\, N_j}_{L'_j}$$

$$\mathcal{R}ep_n G * H \simeq \coprod_{i,j} \underbrace{pt/L_i \underset{pt/GL_n}{\times} pt/L_j}_{\cong\ L_i\backslash GL_n/L_j}$$

dimension monoid for $PSL_2(\mathbb{Z})$. (K. Morrison, 1980 for a general algebra (finitely generated))

① G finite group

$\mathbb{F}_q$ field, $q \wedge |G| = 1$ and $\mathbb{F}_q$ big enough (all simple reps are abs. simple)

$\mathrm{Rep}_G(\mathbb{F}_q) \ni$

Then $M = S_1^{\oplus n_1} \oplus \dots \oplus S_r^{\oplus n_r}$, $S = \{S_1, \dots, S_r\}$ simple reps of G up to iso

$$\dim M \in \mathbb{N}^S =: \Gamma_G$$
$$\dim M = (n_1, \dots, n_r)$$

② $G = PSL_2(\mathbb{Z}) = H * K$ $\quad H = \mathbb{Z}/2\mathbb{Z}$, $K = \mathbb{Z}/3\mathbb{Z}$

$S = \{S_1, \dots, S_r\}$ simples of H

$T = \{T_1, \dots, T_s\}$ simples of K

$M \in \mathrm{Rep}_G(\mathbb{F}_q)$ $\quad \begin{cases} H \hookrightarrow G \\ K \hookrightarrow G \end{cases}$

$$\dim M = \left(\dim \mathrm{Res}^G_H M, \dim \mathrm{Res}^G_K M\right) \in \Gamma_H \times_{\mathbb{N}} \Gamma_K = \Gamma_{H*K}$$

Fact: $\Gamma_H \times_{\mathbb{N}} \Gamma_K \overset{1:1}{\longleftrightarrow}$ connected components of $\mathrm{Rep}_n(H*K)$.

rational fraction count of connected components of $\mathrm{Rep}_n(H*K)$

connected components of $\mathrm{Rep}_n(H*K)$ are of the form

$$L_1 \backslash GL_n / L_2$$ where L_1, L_2 are products of GL_m's

so the stacky number of points is

$$\# L_1 \backslash GL_n / L_2 (\mathbb{F}_q) \simeq \frac{\# GL_n(\mathbb{F}_q)}{\# L_1(\mathbb{F}_q) \, \# L_2(\mathbb{F}_q)} \in \mathbb{Q}(q)$$

main result

Thm: $d \in \Gamma_G$.

$M_{G,d}(q) \in \mathbb{Z}[q]$

$A_{G,d}(q) \in \mathbb{Z}[q]$

$I_{G,d}(q) \in \mathbb{Z}[q]$.

$\in \mathbb{N}[q]$ in fact.

Link between these formulas

plethystic exponential.

$\mathrm{Exp}_z(q^a z^b) = \left(\frac{1}{1-z^b}\right)^{q^a}$

$\mathrm{Exp}_z(q^a z^b) = \frac{1}{1-q^a z^b}$.

$$\sum_{d\in\Gamma_G} M_d(q) z^d = \mathrm{Exp}_z\Big(\sum_{d\in\Gamma_G} I_d(q) z^d\Big) = \mathrm{Exp}_{z,q}\Big(\sum_{d\in\Gamma_G} A_d(q) z^d\Big).$$

Krull-Schmidt

Galois descent for reps of G

$\to$ $(M_d)_d$ family of pols $\Leftrightarrow$ $(I_d)_d$ family of pols $\Leftrightarrow$ $(A_d)_d$ family of pols

$(A_d) \in \mathbb{Z}[q]$ (resp $\mathbb{N}[q]$) $\Rightarrow$ $(M_d) \in \mathbb{Z}[q]$ ($\mathbb{N}[q]$).

- $A_d(q) \in \mathbb{Q}(q) \Rightarrow A_d(q) \in \mathbb{Q}[q]$.
  $\hookrightarrow$ a rat. fract. taking integer values at infinitely many integers is a polynomial.

  (exercise).

- $A_d(q) \in \mathbb{Q}[q] \Rightarrow A_d(q) \in \mathbb{Z}[q]$

  (argument stolen from Bozec-Schiffmann-Vasserot) $\hookrightarrow$ result of Katz in the appendix of Hausel–Rodriguez-Villegas 2008 deep
  if a constructible set has polynomial count, this pol has integer coefficients.

define: $X_d := \{(x,f) \in \mathrm{Rep}_d(G)(\overline{\mathbb{F}_q}) \times GL_n(\overline{\mathbb{F}_q}) \mid f \in \mathrm{Aut}(x)\} \subset \mathrm{Rep}_d(G)(\overline{\mathbb{F}_q}) \times GL_n(\overline{\mathbb{F}_q})$

x is absolutely indecomposable

constructible subset.

$$\#X_d \big/ \#GL_d(\mathbb{F}_q) = A_d(q) \quad \text{by definition.}$$

$\#GL_d(\mathbb{F}_q) \in \mathbb{Z}[q]$ + monic

$A_d(q) \in \mathbb{Q}[q]$

$$\rightsquigarrow \#X_d = A_d(q)\,\#GL_d(\mathbb{F}_q) \in \underbrace{\mathbb{Z}[q]}_{\text{by Katz}}.$$

$$\Rightarrow \quad \#X_d \big/ \#GL_d(\mathbb{F}_q) \in \mathbb{Z}[q].$$

($GL_d(\mathbb{F}_q)$ monic in q)

<u>Goal</u>: $\boxed{A_d(q) \in \mathbb{Q}[q]}$

<u>Steps of the proof</u> (Schiffmann) $d \in \Gamma_G$ $\quad G = PSL_2(\mathbb{Z})$

$I\,Rep_d$ = inertia stack of Rep_d
parametrizes pairs (M, f) of a rep of G and an automorphism of M

$I\,Rep_d^{nil}$ = "nilpotent inertia stack" of Rep_d : f is nilpotent endomorphism of M.

By def, $\quad M_d(q) = \#\,I Rep_d(\mathbb{F}_q)$

"<u>unipotent reduction</u>" $\Rightarrow$

$$\sum_{d \in \Gamma_G} \mathrm{vol}(I Rep_d^{nil}(\mathbb{F}_q))\, z^d = \mathrm{Exp}_{q,z}\left(\sum_{d \in \Gamma_G} \frac{A_d(q)}{q-1} z^d \right)$$

$\Rightarrow$ Suffices to prove $\mathrm{vol}(I Rep_d^{nil}(\mathbb{F}_q)) \in \mathbb{Q}(q)$.

Jordan stratification of $IRep^{nil}_d$

call $|d| = n$. Jordan type $\underbrace{(\alpha_1, \dots, \alpha_s)}_{\alpha}$, $\alpha_i \in \Gamma_G$, $\alpha_1 + \dots + \alpha_s = d$

$(IRep_d^{nil})_\lambda \subset IRep_d^{nil}$ locally closed substack of pairs (M, f) where f is a nilp endo end of M having Jordan type λ

$$(IRep_d^{nil})_\lambda \longrightarrow \prod Rep_{\alpha_i}$$
$$(M, f) \longmapsto \left(\ker\left(\operatorname{im} f^i / \operatorname{im} f^{i+1} \xrightarrow{f} \operatorname{im} f^{i+1} / \operatorname{im} f^{i+2} \right)\right)_i$$

is a "iteration of vector bundle stacks". Ingredient: Rep A is of homological dim one

$\Rightarrow$ fibers have a number of $\mathbb{F}_q$-points of the form q^t for some $t \in \mathbb{Z}$.

- $\# (IRep_d^{nil})_\lambda (\mathbb{F}_q) = q^t \cdot \prod \underset{\mathbb{Q}(q)}{\underset{\cap}{\# Rep_{\alpha_i}(\mathbb{F}_q)}} \in \mathbb{Q}(q)$.

$\Rightarrow \# IRep_d^{nil} \in \mathbb{Q}(q)$ ■

$\boxed{\text{Positivity:}}$ can be deduced of the purity of the representation stacks via rather intricate arguments due to (Davison-Meinhardt)

Prop The representation stacks $Rep_n(A)$ have pure compactly supported cohomology

Proof: Connected components of $Rep_n(A)$ are of the form $L_1 \backslash Gl_n / L_2$ where L_1, L_2 are levi subgroups of Gl_n.

$P_2 \leftrightarrow L_2$ parabolic.

$Gl_n/L_2 \longrightarrow Gl_n/P_2$ is a loc trivial affine fiber over a sm. proj var.

$\Rightarrow Gl_n/L_2$ is pure

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- L_q is a product of Gl_n's, so $H_{L_q}(pt)$ is pure

$\Longrightarrow$ $H_{c,L_q}(G/L_2)$ is pure.

+ existence of BPS Lie algebra.

Examples: in dim 1: reps of $\mathbb{Z}/2 \times \mathbb{Z}/3$.

For $d \in (P_{PSL_2(\mathbb{Z})})_1$, $A_d(q) = 1$

in dim 2:
counting reps M of $PSL_2(\mathbb{Z})$ s.t. $Res^M_{\mathbb{Z}/2} \simeq \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$ ξ prim 2nd root of 1

$Res^M_{\mathbb{Z}/3} \simeq \begin{pmatrix} 1 & 0 \\ 0 & \xi' \end{pmatrix}$ ξ' prim 3rd root of 1

d associated dim vector

$$\rightarrow \frac{\left(\frac{(q^2-1)(q^2-q)}{(q-1)^2} - 2\right)}{q-1} + 2$$

$$= \frac{q(q+1)-2}{q-1} + 2$$

$$= q+4$$

so $M_d(q) = q+4$
and $A_d(q) = q+2$

- $G = \mathfrak{S}_3 * \mathfrak{S}_3$ satisfies all properties to have polynomiality of the point count of representations but
  - purity of the stack fails
  - and positivity fails.

→ in dim 2, simple representation of $\mathfrak{S}_3$, S

- ⇒ a connected component of $\mathrm{Rep}_2 G$ is $\mathbb{C}^*\backslash GL_2/\mathbb{C}^* \simeq \underbrace{B\mathbb{C}^*}_{\text{pure}} \times \underbrace{PGL_2}_{\text{impure}}$.

- $d = ([S], [S]) \in \Gamma_{\mathfrak{S}_3 * \mathfrak{S}_3}$

$$M_d(q) = \frac{(q^2-1)(q^2-q)}{q-1} = q(q^2-1). \quad \underline{\text{not positivity.}}$$

- $G = H * A$ A abelian group and H arbitrary, will satisfy positivity.

Complements

1) "nilpotent reduction"

$$\sum_{d\in\Gamma_G} \mathrm{vol}\left(\mathfrak{I}\mathrm{Rep}_d^{\mathrm{nil}}(\mathbb{F}_q)\right) z^d = \mathrm{Exp}_{q,z}\left(\sum_{d\in\Gamma_G} \frac{A_d(q)}{q-1} z^d\right)$$

2) $$\left(\mathfrak{I}\mathrm{Rep}_d^{\mathrm{nil}}(G)\right)_{\underline{\alpha}} \longrightarrow \prod_{i=1}^{s} \mathrm{Rep}_{\alpha_i} G$$

is an iteration of vector bundle stacks.

⚠ Formula

$$R_7^{\mathrm{iso},\mathcal{Y}}(q) = \#\text{ isoclasses of rep of } PSL_2(\mathbb{Z}) \text{ of dim } 7$$
$$= 6q^8 + 24q^7 + 84q^6 + 294q^5 + 1014q^4 + 3192q^3 + 8640q^2 + 16812q + 18882$$