Hodge Club - 29 October 2021

Categorification of Hall algebras

I - General formalism

II - 1D CoHA of a quiver

III - The constructible Hall algebra of a finitary category

IV - Perverse sheaves categorification of quantum groups.

I - General formalism

$\mathcal{A}$ exact or abelian category

$\tilde{\mathcal{A}}$ groupoid of objects in $\mathcal{A}$.

$\rightarrow$ category with the same objects as $\mathcal{A}$ but we keep only isomorphisms among morphisms.

$\mathcal{A}^{(1)}$ category of exact sequences in $\mathcal{A}$

objects : $0 \to N \xrightarrow{a} E \xrightarrow{b} M \to 0$ (pairs of morphisms $N \xhookrightarrow{a} E$, $E \xrightarrow{b} M$)

morphisms
$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "a"] \arrow[d, "f"'] & E \arrow[r, "b"] \arrow[d, "g"'] & M \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & N' \arrow[r, "a'"'] & E' \arrow[r, "b'"'] & M' \arrow[r] & 0 \end{tikzcd}$$

$\tilde{\mathcal{A}}^{(1)}$ groupoid of exact sequences.

Natural correspondence :

$$\begin{tikzcd} & 0 \to N \to E \to M \to 0 \arrow[dl] \arrow[dr] & \\ (M,N) & \mathcal{A}^{(1)} \arrow[dl, "q"'] \arrow[dr, "p"] & E \\ \mathcal{A} \times \mathcal{A} & & \mathcal{A} \end{tikzcd}$$

\begin{tikzcd}
 & \tilde{\mathcal{A}}^{(n)} \arrow[dl, "\tilde{q}"'] \arrow[dr, "\tilde{p}"] & \\
\tilde{\mathcal{A}} \times \tilde{\mathcal{A}} & & \tilde{\mathcal{A}}
\end{tikzcd}

The fiber of $\tilde{q}$ over (M,N) is $\mathrm{Ext}^1(M,N) / \mathrm{Hom}(M,N)$.

~~fiber product of groupoids~~ (tells you also how to define the fiber product of stacks)

\begin{tikzcd}
\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "G"] \\
\mathcal{Z} \arrow[r, "F"'] & \mathcal{Y}
\end{tikzcd}

$\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is the groupoid with objects triples

(z, x, f) $\quad$ $z \in$ object of $\mathcal{Z}$

$x \in \mathrm{ob}(\mathcal{X})$

$f \in \mathrm{Isom}(F(z), G(x))$

and a morphism of triples $(z, x, f) \xrightarrow{g} (z', x', f')$ is a pair $(g_z, g_x) \in \mathrm{Isom}(z, z') \times \mathrm{Isom}(x, x')$ such that

\begin{tikzcd}
F(z) \arrow[r, "F(g_z)"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & F(z') \arrow[d, "f'"] \\
G(x) \arrow[r, "G(g_x)"'] & G(x')
\end{tikzcd}

<u>examples</u> $y \in \mathcal{Y}$ object

$$\begin{tikzcd} \mathcal{X}_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{X} \arrow[d, "G"] \\ y \arrow[r] & \mathcal{Y} \end{tikzcd}$$

$\mathcal{X}_y = \{(x, f) \quad y \xrightarrow[f]{\sim} G(x)\}$ + morphisms.

e.g. if $\mathcal{X} = pt/H$, $\mathcal{Y} = pt/G$, $H \to G$ group morphism

then

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & pt/H \arrow[d] \\ pt \arrow[r] & pt/G \end{tikzcd}$$

$ob(*) = G$

$g, g' \in G$.

$$\begin{tikzcd} * \arrow[r, "g"] \arrow[dr, "g'"'] & * \arrow[d, "h"] \\ & * \end{tikzcd}$$

$Hom(g, g') = \{h \in H, \ hg = g'\}$

$\longrightarrow * = G/H$.

Very often (if not always), we can upgrade everything to stacks:

$\mathfrak{X}_{\mathcal{A}}$ stack of objects in $\mathcal{A}$

$\mathfrak{X}_{\mathcal{A}^{(1)}}$ stack of short exact sequences.

$$\begin{tikzcd} & \mathfrak{X}_{\mathcal{A}^{(1)}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathfrak{X}_{\mathcal{A}} \times \mathfrak{X}_{\mathcal{A}} & & \mathfrak{X}_{\mathcal{A}} \end{tikzcd}$$

The complexity of the geometry of q is directly related to the homological dimension of $\mathcal{A}$.

examples to have in mind ① $Q=(I,\Omega)$ quiver

$\mathcal{A} = \mathrm{Rep}_Q(k)$ representations of a quiver over a field k.

$= \bigsqcup_{d \in \mathbb{N}^I} (E_d / G_d)$ stack quotient, or sometimes groupoid. but never set-theoretic quotient.

$$E_d = \bigoplus_{i \to j \in \Omega} \mathrm{Hom}(k^{d_i}, k^{d_j})$$

$$G_d = \prod_{i \in I} GL_{d_i}$$

or maybe $\mathcal{A} = \mathrm{Rep}_{\Pi_Q}(k)$ Π_Q = preprojective algebra of Q

or maybe $\mathcal{A} = \mathrm{Rep}_{\mathrm{Jac}(Q,W)}(k)$, (Q,W) quiver with potential and $\mathrm{Jac}(Q,W)$ its Jacobi algebra.

or $\mathcal{A} = \mathrm{Coh}(C)$ for C smooth projective curve

or $\mathcal{A} = \mathrm{Higgs}(C)$

or $\mathcal{A} = \mathrm{Coh}_c(\mathrm{Tot}(\omega_C \oplus \mathcal{O}_C))$.

or $\mathcal{A} = \mathrm{Coh}_c(S)$, S quasiprojective 2-CY surface (except for $S = T^*C$, C sm. pr. curve, this is very far from being developed)

We need to associate to the stacks / groupoids $\mathcal{Y}$ under consideration a space $F(\mathcal{Y})$ s.t we can define a pull-back q^* and a push-forward p_*.

Then, $p_* q^* : F(\mathcal{M}_{\mathcal{A}}) \times F(\mathcal{M}_{\mathcal{A}}) \longrightarrow F(\mathcal{M}_{\mathcal{A}})$

gives an associative multiplication on $F(\mathcal{M}_{\mathcal{A}})$.

What can be F ?

* If $k = \mathbb{F}_q$ is a finite field (and $\mathcal{A}$ is $\mathbb{F}_q$-linear), we can consider

$$\mathrm{Fun}_c\left(\mathcal{M}_{\mathcal{A}}(\mathbb{F}_q), \mathbb{C}\right)$$

finite support

* We can consider H^* cohomology
* We can consider K-theory
* ~~We~~ The stack ~~(s)~~ under consideration has(n) more structure, for example a "d-critical structure", on can consider the vanishing cycle cohomology.

Actually, the simplest situation is the one with $\mathrm{Rep}_Q(\mathbb{C})$ and cohomology of the moduli spaces.

1D - CoHA of a quiver [Kontsevich-Soibelman] "additive formal group law"

$Q = (I, \Omega)$ quiver

$$\mathcal{M}_d = \bigsqcup_{d \in \mathbb{N}^I} E_d / G_d$$

$$\mathrm{CoHA}(Q) := H^*(\mathcal{M}_d) = \bigoplus_{d \in \mathbb{N}^I} \underbrace{H^*(E_d/G_d)}_{}$$

$$= H^*_{G_d}(E_d)$$

$$= H^*_{G_d}(pt)$$

$$= \mathbb{C}\left[x_{i,j} \,,\, \begin{matrix} i \in I \\ 1 \leq j \leq d_i \end{matrix}\right]^{\mathfrak{S}_d}$$

$\mathfrak{S}_d = \prod_{i \in I} \mathfrak{S}_{d_i}$ product of symmetric groups.

product : This is a shuffle algebra product.

Simpler situation

$$H = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[x_{n1}, \dots, x_{nn}]^{\mathfrak{S}_n}$$

How to build a graded multiplication on H?

$g \in \mathbb{C}(x)$

$$f(x_{n1}, \dots, x_{nn}) \cdot g(x_{m1}, \dots, x_{mm}) = \frac{1}{n!\,m!} \mathrm{Sym}\left(f(x_{n1} \dots x_{nn}) g(x_{m1}, \dots, x_{mm}) \prod_{k=1}^{n} \prod_{l=1}^{m} g(x_{nk} - x_{ml}) \right)$$

g-weighted shuffle product.

This is the kind of multiplication we obtain on $\mathrm{CoHA}(Q)$,

$f \in \mathrm{CoHA}(Q)[d]$

$g \in \mathrm{CoHA}(Q)[e]$

$$f(x_{ij})\, g(y_{ij}) = \frac{1}{d!\,e!} \mathrm{Sym}\left(f(x_{ij})\, g(y_{ij}) \frac{\prod_{u,v \in I} \prod_{k=1}^{d_u} \prod_{l=1}^{e_v} (y_{vl} - x_{uk})^{a_{uv}}}{\prod_{u \in I} \prod_{k=1}^{d_u} \prod_{l=1}^{e_u} (y_{ul} - x_{uk})} \right)$$

$$a_{uv} = \#\{u \to v \in Q\}.$$

Recent developments : Vertex $^{(co)}$algebra structure on $\mathrm{CoHA}(Q)$ + compatibility avec algebra structure

Explanation of the computation of the product for $Q = \bullet$

(computation using basic facts on equivariant cohomology)

$G \circlearrowright X$

reductive group

$H_G^*(X)$ is a $H_G^*(pt)$-module, using

the map $p^* : H_G^*(pt) \longrightarrow H_G^*(X)$; $p: X \longrightarrow pt$ G-equivariant. $\mathfrak{S}_n$.

$X = pt$; $T = (\mathbb{C}^*)^n$ $H_T^*(pt) = \mathbb{C}[x_1, \dots, x_n]$; $G = Gl_n$; $H_G(pt) = \mathbb{C}[x_1, \dots, x_n]$

Theorem: ① $T \subset G$ maximal torus, $W = N_G(T)/T$ Cartan.

$$H_G^*(X) \simeq H_T^*(X)^W$$

② $X \circlearrowleft T$ torus

X^T fixed points

$X^T \overset{i}{\hookrightarrow} X$ inclusion of fixed points

pull-back in equivariant cohomology

$$i^* : H_T^*(X) \longrightarrow H_T^*(X^T) = H^*(X^T) \otimes H_T^*(pt)$$

push-forward : $i_* : H_T^*(X^T) \longrightarrow H_T^*(X)$

become isomorphism in localized equivariant cohomology :

$$i^* : H_T^*(X) \otimes_{H_T^*(pt)} \mathrm{Frac}(H_T^*(pt)) \xrightarrow{\sim} H^*(X^T) \otimes \mathrm{Frac}\, H_T^*(pt).$$

and $i^* i_* \in \mathrm{End}\left(H_T^*(X) \otimes_{H_T^*(pt)} \mathrm{Frac}\, H_T^*(pt)\right)$ is multiplication by the equivariant Euler class $e(\nu_{X^T})$ of X^T the fixed point locus.

We have the tools to compute the multiplication of $CoHA(\bullet)$.

If G is a group, $BG = pt$ with action of $G = pt/G$.

$$Q = (\bullet, \emptyset)$$

$d \in \mathbb{N}$, $d'+d''=d$

$\mathcal{M}_d = BGL_d$

$\mathcal{M}^{(1)}_{d',d''} = BP_{d',d''}$

(standard)

$P_{d',d''} \subset GL_{d'+d''}$ parabolic, stabilizer of a d'-dimensional subspace of $\mathbb{C}^{d'+d''}$.

correspondence

$$\begin{tikzcd}
 & & \bigsqcup_{i_1<\dots<i_d} B\mathbb{G}_m^{d'+d''} \arrow[ddll, "a"'] \arrow[d, "i"] \arrow[ddr, "c"] & \\
 & & P_{d',d''}\backslash GL_{d'+d''}/\mathbb{G}_m^{d'+d''} \arrow[dl, "f"'] \arrow[dr, "e"] & \\
B\mathbb{G}_m^{d} \times B\mathbb{G}_m^{d'} \arrow[d, "b"'] & BP_{d',d''} \arrow[dl, "q"'] \arrow[dr, "p"] & & B\mathbb{G}_m^{d'+d''} \arrow[dl, "d"] \\
BGL_{d'} \times BGL_{d''} & & BGL_{d'+d''} &
\end{tikzcd}$$

$\mathbb{G}_m^{d'+d''} \circlearrowright P_{d',d''}\backslash GL_{d'+d''}$

by

$$t\cdot\left(P_{d',d''}\cdot g\right) = P_{d',d''}\cdot g t^{-1}$$
$$= P_{d',d''}\, t g t^{-1}$$

since $t \in P_{d',d''}$.

To understand $p_* q^*$ acting on cohomology, it suffices to understand

$$d^* p_* q^*$$
$$= e_* f^* q^*$$
$$= \underbrace{e_* \frac{1}{e(N_i)} i_*}_{c_*} \underbrace{i^* f^* q^*}_{= a^* b^*}$$

$\rightsquigarrow$ split the calculation on each component of $\bigsqcup_{i_1<\dots<i_{d'}} B\mathbb{G}_m^{d'+d''}$.

$\rightarrow$ fixed points are given $F \subset \mathbb{C}^{d'+d''}$

$e_1, \dots, e_{d'+d''}$ base de $\mathbb{C}^{d'+d''}$

$F = \langle e_{i_1}, \dots, e_{i_d} \rangle$

for $\{i_1 < \dots < i_d\} \subset \{1, \dots, d'+d''\}$.

On $B\mathbb{G}_m^{d'+d''}$ corresponding to $i_1 < \dots < i_{d'}$, $p \in {}^{GL_{d'+d''}}\backslash_{P_{d',d''}}$ corresponding

$N_i|_p \simeq T_p \, P_{d',d''}\backslash {}^{GL_{d'+d''}}$ fixed point.

The weights of $\mathbb{G}_m^{d'+d''}$ on this space are

$$\alpha_i - \alpha_j$$

for $i \in \{1,\dots,d'+d''\} \setminus \{i_1,\dots,i_{d'}\} = [\![1, d'+d'']\!] \setminus I$

$j \in \{i_1, \dots, i_{d'}\} = I$

So $e(N_i)|_p = \prod_{i \in [\![1,d'+d'']\!] \setminus I} \prod_{j \in I} (x_i - x_j) \in H^*_{\mathbb{G}_m^{d'+d''}}(pt)$.

Altogether, we obtain the desired formula:

$$\boxed{f(x_1,\dots,x_{d'}) * g(x_{d'+1},\dots,x_{d'+d''}) = \sum_{\substack{I \sqcup J = \{1,\dots,d'+d''\} \\ |I| = d'}} f(x_I)\, g(x_J) \frac{1}{\prod_{i\in J}\prod_{j \in I}(x_i - x_j)}}.$$

Next simple situation:

$Q = (I, \Omega)$ quiver and K-theory

$$KHA(Q) = \bigoplus_{d \in \mathbb{N}^I} \underbrace{K(E_d / G_d)}$$

$$\shortparallel$$

$$K^{G_d}(pt)$$

$$\shortparallel$$

$$\mathbb{Z}[x_{i,k}^{\pm 1},\ i \in I,\ 1 \leq k \leq d_i]^{\mathfrak{S}_d}.$$

Product: "multiplicative formal group law".

$f \in KHA(Q)[d]$, $g \in KHA(Q)[e]$.

$$f(x_{i,k})\, g(y_{ik}) = \frac{1}{d!\,e!} \mathrm{Sym}\left(f(x_{ik}) g(y_{ik}) \frac{\prod_{u,v \in I} \prod_{k=1}^{d_u} \prod_{l=1}^{e_v} \left(y_{vl}/x_{uk}\right)^{a_{uv}}}{\prod_{u \in I} \prod_{k=1}^{d_u} \prod_{l=1}^{e_u} y_{ul}/x_{uk}} \right).$$

Tricky situation: elliptic cohomology

Guess what the formula is.

$$\mathrm{Ell}(E_d/G_d) = \mathrm{Ell}_{G_d}(pt)$$

$$= (E^d)^{\mathfrak{S}_d}.$$

$\mathcal{L}, \mathcal{L}'$ line bundles on $(E^d)^{\mathfrak{S}_d}, (E^{d'})^{\mathfrak{S}_{d'}}$

$$\begin{tikzcd} & F_{dd'}/P_{d,d'} \arrow[dl] \arrow[dr] & \\ E_d/G_d \times E_{d'}/G_{d'} & & E_{d+d'}/G_{d+d'} \end{tikzcd}$$

f, f' sections

$f \boxtimes f' \cdot \prod_{\dots} \vartheta(\dots)$ section l.b

Constructible functions on groupoid of $\mathbb{F}_q$-points

→ Constructible Hall algebra, as originally defined by Ringel for quivers.

$\mathcal{A}$ is a finitary category

i.e. $\forall M, N$ objects of $\mathcal{A}$, $\mathrm{Ext}^\bullet_{\mathcal{A}}(M,N)$ is a finite set.

Then one can define an associative product on

$$H_{\mathcal{A}} = \bigoplus_{[M]\in ob(\mathcal{A})/\sim} \mathbb{C}\cdot[M] = \mathrm{Fun}_c\left(ob(\mathcal{A})/\sim, \mathbb{C}\right)$$

in the following way

$$[M]*[N] = \underbrace{\overbrace{\sqrt{\frac{|\mathrm{Hom}(M,N)|}{|\mathrm{Ext}^1(M,N)|}}}^{=:\,\langle M,N\rangle_m \text{ "multiplicative Euler form".}}}_{\text{to makes the product have more pleasant properties.}} \sum_{[R]\in ob(\mathcal{A})/\sim} \underbrace{\#\{N'\subset R \mid N'\simeq N \;\&\; R/N'\simeq M\}}_{F^R_{MN}} [R]$$

Associativity: not hard to check.

One can build a comultiplication in the dual way:

$$\Delta([R]) = \sum_{[M],[N]} \sqrt{\langle M,N\rangle_m}\, \frac{a_M\, a_N}{a_R}\cdot [M]\otimes[N]$$

Green's thm : If $\mathcal{A}$ is hereditary ($\mathrm{Ext}^i(-,-)=0$ for $i \geq 2$) then $H_{\mathcal{A}}$ is a bialgebra :

$$\Delta \circ m = m \otimes m \circ \underbrace{\sigma_{23}}_{\text{swap of } 2^{nd} \,\&\, 3^{rd} \text{ factors.}} \circ \Delta \otimes \Delta$$

$Q = (I, \Omega)$ quiver ; $\mathbb{F}_q$ finite field

$\mathcal{A} = \mathrm{Rep}_Q(\mathbb{F}_q)$

$H_{\mathcal{A}} =: H_{Q,\mathbb{F}_q}$ Hall algebra of Q over $\mathbb{F}_q$.

Structure of $H_{Q,\mathbb{F}_q}$?

Ringel's theorem

Assume Q loop free (for simplicity)

$U_\nu(\mathfrak{n}^+)$ quantum unipotent enveloping algebra.

$E_i \longmapsto [S_i]$ induces an injective algebra morphism

$$U_{\sqrt{q}}(\mathfrak{n}^+) \xrightarrow{\phi} H_{Q,\mathbb{F}_q}.$$

S_i simple representation of Q

$\dim S_i = i \in \mathbb{N}^I$

$E_i, i \in I$: Chevalley generators.

$U_v(n^+)$ is the associative algebra generated by $E_i, i \in I$ subject to the Serre relations

$$\sum_{l=0}^{1-a_{ij}} \binom{1-a_{ij}}{l}_q E_i^{l} E_j E_i^{1-a_{ij}-l} = 0$$

$\forall i \neq j$

where $(a_{ij}) = A$ is the Cartan matrix of Q:

$a_{ij} = 2\delta_{ij} - \#\{\alpha : i \to j\} - \#\{\alpha : j \to i\}$.

★ ϕ surjective $\iff$ Q is of finite type

<u>example</u>: $Q = \bullet$

$Rep_Q(\mathbb{F}_q) = Vect_{\mathbb{F}_q}$ Euler form: $\langle m,n \rangle_m = q^{mn}$.

objects $= \mathbb{N}$ integers

$H_{Q,\mathbb{F}_q} = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[n]$

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \frac{[m+n]_q!}{[m]_q! \, [n]_q!}$$

$$[m]\cdot[n] = \underbrace{\begin{bmatrix} m+n \\ m \end{bmatrix}_q \sqrt{q^{mn}}}\, [m+n]$$

$$[m]_q! = \prod_{i=1}^{m} [i]_q!$$

$$[i]_q = \frac{q^n - 1}{q-1}$$

$$= \binom{m+n}{m}_{\sqrt{q}} [m+n]$$

symmetrized version of q-binomial coefficients.

Categorification of the quantum group $U_v(\mathfrak{n}^+)$

(Lusztig)

Instead of functions on $\mathcal{M}_Q(\mathbb{F}_q)$, consider perverse sheaves on $\mathcal{M}_Q$.

constructible

constructible

The base field can be $\mathbb{C}$ or $\overline{\mathbb{F}_q}$.

$d'+d''=d$, $\quad d, d', d'' \in \mathbb{N}^I$.

$$\begin{tikzcd}[column sep=small]
 & & \{0\to N\to R\to M\to 0\} \arrow[dll, mapsto] \arrow[drr, mapsto] & & \\
 & & \mathcal{M}^{(1)}_{d',d''} \arrow[dl, "q"'] \arrow[dr, "p"] & & \\
(M,N) & \mathcal{M}_{d'}\times\mathcal{M}_{d''} & & \mathcal{M}_d & R
\end{tikzcd}$$

Remind that q is smooth (of relative dimension $-\langle d', d''\rangle$) & p is proper.

Define

$$\mathrm{Ind}_{d',d''} : D^b_c(\mathcal{M}_{d'}\times\mathcal{M}_{d''}) \longrightarrow D^b_c(\mathcal{M}_d)$$

$$\mathcal{F} \longmapsto p_* q^*[-\langle d', d''\rangle]$$

Takes semisimple complexes to semisimple complexes

Lusztig considers $\underbrace{\text{the category}}_{\text{(additive)}}$ $\mathcal{Q}$ $\underbrace{\text{of constructible}}_{\text{(semisimple)}}$ complexes on $\mathcal{M} = \bigsqcup_{d \in \mathbb{N}^I} \mathcal{M}_d$, containing

* $\mathbb{C}_{\mathcal{M}_i}$ for $i \in I$,
* stable under shifts, direct summands
* stable under $\mathrm{Ind}_{d',d''}$.

$\mathcal{P} \subset \mathcal{Q}$

("semisimple) perverse sheaves contained in $\mathcal{Q}$.

$K_{\oplus}(\mathcal{Q})$ becomes an algebra, the product being induced by $\mathrm{Ind}_{d',d''}$.

It is a $\mathbb{Z}[v^{\pm 1}]$-algebra where v acts as shift $[1]$.

<u>Thm</u> (Lusztig) The map

$$\mathcal{U}_{v^{-1}}^{\mathbb{Z}}(n^+) \longrightarrow K_{\oplus}(\mathcal{Q})$$
$$E_i \longmapsto \mathbb{C}_{\mathcal{M}_i}[\dim \mathcal{M}_i]$$

is well defined and induces an isomorphism of algebras.

$U_v^{\mathbb{Z}}(n^+)$ is Lusztig integral form of $U_v(n^+)$.

In other words, $\mathcal{Q}$ categorifies $U_v^{\mathbb{Z}}(n^+)$.

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* If one works over $\overline{\mathbb{F}_q}$, there is an other way to decategorify the category $\mathcal{Q}$.
* Over $\mathbb{C}$, one can consider the characteristic cycle of constructible sheaves.

But this is still a long story.