Exercise sheet 6

Thursday, 17 December 2020

Exercise 6.1. Borel subalgebras.

1. Show that any Borel Lie subalgebra of a finite dimensional Lie algebra is its own normalizer.

2. Show that a Borel subalgebra contains the radical.

Exercise 6.2. Lie bialgebras and Poisson-Lie groups A Poisson algebra is a the datum $(A, \{-,-\})$ of a commutative associative algebra A and $\{-,-\} : A^{\otimes 2} \to A$ is a Poisson bracket, that is a Lie bracket satisfying the Leibniz rule: for any $a, b, c \in A$,

$$\{ab, c\} = a\{b, c\} + \{a, c\}b.$$

A Poisson manifold is a smooth manifold M with a Poisson structure on $\mathcal{C}^\infty(M)$.

1. Let M be a Poisson manifold. Show that there exists a Poisson bivector field $\Pi \in \Gamma(\Lambda^2 TM)$ such that for any $f, g \in \mathcal{C}^\infty(M)$, $\{f, g\} = df \otimes dg(\Pi)$.

2. Express the Jacobi identity as a property verified by Π.

3. Show that any symplectic manifold has a natural Poisson structure.

4. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\mathfrak{g}^*$ the dual. Define

$$\{f, g\}(x) = x([df(x), dg(x)])$$

for $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$. Show that this endows $\mathfrak{g}^*$ with a Poisson structure.

5. Let M and N be Poisson manifolds. Define a natural Poisson structure on the product $M \times N$.

We introduce now Poisson-Lie groups. A Poisson manifold G endowed with the structure of a Lie group is called a Poisson-Lie group if the multiplication

$$m : G \times G \to G$$

is a Poisson map.

6. Express the fact that m is a Poisson map using the comultiplication

$$\Delta : \mathcal{C}^\infty(G) \to \mathcal{C}^\infty(G \times G)$$

7. Let Π be the Poisson bivector field associated to the Poisson structure on G. Express that G is a Poisson-Lie group by a condition verified by Π. Compute $\Pi(e)$ where $e \in G$ is the identity element.

8. Is the inversion map $i : G \to G$ a Poisson map?

We now come to Lie bialgebras.

9. Let M be a Poisson manifold with Poisson bracket $\{-,-\}$. Let $\Pi \in \Gamma(\Lambda^2 TM)$ be the corresponding Poisson bivector. Assume that $\Pi(x_0) = 0$ for some $x_0 \in M$. Show that the formula

$$\begin{array}{cccc} [-,-] & : & T^*_{x_0}M \otimes T^*_{x_0}M & \to & T^*_{x_0}M \\ & & df(x_0) \otimes dg(x_0) & \mapsto & d(df \otimes dg(\Pi))(x_0) \end{array}$$

endows $T^*_{x_0}M$ with a structure of a Lie algebra.

When G is a Poisson-Lie group, we obtain a Lie bracket $[-,-] : \Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$. The dual map

$$\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$$

is called the cobracket.

10. Express the Jacobi identity of the Lie bracket of $\mathfrak{g}^*$ as a property satisfied by δ. This is called the coJacobi identity. A couple $(\mathfrak{a}, \delta)$ where δ is a Lie cobracket satisfying the coJacobi identity is called a Lie coalgebra.

11. Describe δ in terms of the Poisson bivector.

12. Show that for any $a, b \in \mathfrak{g}$,

$$\delta([a,b]) = [\delta(a), 1 \otimes b + b \otimes 1] + [a \otimes 1 + 1 \otimes a, \delta(b)]$$

This is the cocycle condition.

A Lie bialgebra is a triple $(\mathfrak{g}, [-,-], \delta)$ such that $(\mathfrak{g}, [-,-])$ is a Lie algebra, δ a Lie cobracket satisfying the coJacobi identity and the cocycle condition. What we just proved is that the tangent space at the identity of a Poisson-Lie group has the structure of a Lie bialgebra.

13. Zeroth example. Let $\mathfrak{g}$ be a finite dimensional LIe algebra. Show that $\mathfrak{g}^*$ has the structure of a Poisson-Lie group.

14. A first example. Recall that up to isomorphism, there is a unique nonabelian Lie algebra of dimension 2. Describe all the Lie bialgebra structure on it.

15. We consider $\mathfrak{sl}_2$ with its usual basis e, f, h Show that the following formulas define a Lie bialgebra structure on $\mathfrak{sl}_2(\mathbf{C})$:

$$\delta(e) = \frac{1}{2} e \wedge h, \quad \delta(f) = \frac{1}{2} f \wedge h, \quad \delta(h) = 0.$$

The first main theorem of the theory of Poisson-Lie groups and Lie bialgebras, due to Drinfeld, states that the functor $F(G) = Lie(G)$ from the category of simply connected Poisson-Lie groups to the category of finite dimensional Lie bialgebras is an equivalence of categories.

Reference: Etingof, Schiffmann, *Lectures on Quantum Groups.*