Exercise sheet 5

Thursday, 10 December 2020

Recall that a finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if its radical (the maximal solvable ideal) is 0.

Exercise 5.1.

1. Show that a finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if it has no nonzero abelian ideal.

2. Show that the root system $\Delta \subset \mathfrak{h}^*$ of a semisimple Lie algebra generates $\mathfrak{h}^*$.

Exercise 5.2. A semisimplicity criterion. Let $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha\right)$ where $\Delta \subset \mathfrak{h}^*$ is a finite subset be a decomposition of a finite dimensional Lie algebra into a direct sum of subspaces such that:

1. $\mathfrak{h}$ is an abelian subalgebra; $\dim \mathfrak{g}_\alpha = 1$ for any $\alpha \in \Delta$, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, a] = \alpha(h) \text{ for any } h \in \mathfrak{h}\}$,
2. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbf{C}h_\alpha$ for some $h_\alpha \in \mathfrak{h}$ such that $\alpha(h) \neq 0$,
3. $\mathfrak{h}^*$ is spanned by Δ.

Show that $\mathfrak{g}$ is a semisimple Lie algebra.

Exercise 5.3. Indecomposable root systems. Let $\Delta \subset \mathfrak{h}^*$ be a finite subset. We say it is indecomposable if it cannot be written $\Delta = \Delta_1 \sqcup \Delta_2$ with Δ_i non-empty and for any $\alpha \in \Delta_1$, $\beta \in \Delta_2$, $\alpha + \beta \notin \Delta \cup \{0\}$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the inclusion of a Cartan subalgebra of a semisimple Lie algebra and let $\Delta \subset \mathfrak{h}$ be its root system.

1. Show that $\mathfrak{g}$ is simple if Δ is indecomposable. (The converse is also true, but we need to know a bit more about root systems).

2. Show that $\Delta \subset \mathfrak{h}^* \setminus \{0\}$ is indecomposable if and only if for any $\alpha, \beta \in \Delta$, there exists $s \geq 1$ and $\gamma_1, \ldots, \gamma_s \in \Delta$ with $\gamma_1 = \alpha$, $\gamma_s = \beta$ and for any $1 \leq i \leq s-1$, $\gamma_i + \gamma_{i+1} \in \Delta \cup \{0\}$.

Exercise 5.4. Orthogonal Lie algebras. Let V be a finite dimensional vector space and $B \in (V \otimes V)^*$ a bilinear form.

1. Show that

$$\mathfrak{o}_{V,B} = \{a \in \mathfrak{gl}(V) \mid B(au, v) + B(u, av) = 0 \text{ for all } u, v \in V\}$$

is a sub-Lie algebra of $\mathfrak{gl}_n(\mathbf{C})$.

2. If M is the matrix of the bilinear form B in some fixed basis of V, show that it can be identified with

$$\mathfrak{o}_{n,M} = \{a \in \mathfrak{gl}_n(\mathbf{C}) \mid a^T M + Ma = 0\}.$$

From now on, M is assumed to symmetric and nondegenerate.

3. Show that the Lie algebra $\mathfrak{o}_{n,M}$ does not depend on M (up to isomorphism).

4. Let M be the antidiagonal unit matrix of size $n \times n$ (the matrix which becomes the unit matrix in a mirror). The corresponding Lie algebra is denoted $\mathfrak{so}_n(\mathbf{C})$. Describe it more explicitly.

5. What is $\mathfrak{so}_2$? Is it simple?

6. Show that for $n \geq 3$, $\mathfrak{so}_n$ is semisimple using the criterion given by Theorem 4.2.

7. Prove that $\mathfrak{so}_n$ is simple if $n = 3$ or $n \geq 5$ by showing that Δ is indecomposable. It is a good idea to treat the case $n > 5$ and the case $n = 5$ afterwards.

8. Show that

$$\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_1\} \sqcup \{\epsilon_1 + \epsilon_2, -\epsilon_1 - \epsilon_2\}$$

is the decomposition of the root system Δ of $\mathfrak{so}_4$ into indecomposable root systems. Deduce that $\mathfrak{so}_4$ is isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Give an explicit isomorphism.

Exercise 5.5. Symplectic Lie algebras. In the context of Exercise 4.4, we take for the bilinear form B any nondegenerate skew-symmetric bilinear form. In the matrix description, we can choose M to be the antidiagonal matrix whoose image in a mirror is diagonal with blocks $I_n, -I_n$. The corresponding Lie algebra is $\mathfrak{sp}_{2n}(\mathbf{C})$.

1. Show that

$$\mathfrak{sp}_{2n} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \text{ are } n \times n \text{ matrices such that } b = b', c = c', d = -a' \right\}$$

where a' denotes the transpose of a with respect to the antidiagonal, similarly for b, c, d.

2. Show that $\mathfrak{sp}_{2n}$ is simple for all $n \geq 1$.

We now understand quite explicitly the root systems of the classical simple Lie algebras.

1. A_n: $\mathfrak{sl}_n$, $n \geq 1$
2. B_n: $\mathfrak{so}_{2n+1}$, $n \geq 1$
3. C_n: $\mathfrak{sp}_{2n}$, $n \geq 1$
4. D_n: $\mathfrak{so}_{2n}$, $n \geq 4$

These families are not completely disjoint. There are exceptional isomorphisms.

The exceptional Lie algebras are

1. G_2: $\mathfrak{g}_2$, $\dim \mathfrak{g}_2 = 14$, $\mathrm{card}\Delta_{G_2} = 12$
2. F_4: $\mathfrak{f}_4$, $\dim \mathfrak{f}_4 = 52$, $\mathrm{card}\Delta_{G_4} = 48$
3. E_6, E_7, E_8: $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, $\dim \mathfrak{e}_6 = 78$, $\dim \mathfrak{e}_7 = 133$, $\dim \mathfrak{e}_8 = 248$, $\mathrm{card}\Delta_{E_6} = 72, \mathrm{card}\Delta_{E_7} = 126, \mathrm{card}\Delta_{E_8} = 240$