
Exercise sheet 4

Thursday, 3 December 2020

Exercice 4.1. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two semisimple Lie algebras. Show that any extension

$$0 \to \mathfrak{g}_1 \to \mathfrak{g} \to \mathfrak{g}_2 \to 0$$

is trivial.

Exercice 4.2. Let $\mathfrak{g}$ be a Lie algebras and $\mathrm{Der}(\mathfrak{g})$ the Lie algebra of derivations. Show that $\mathrm{ad}(\mathfrak{g})$ is an ideal.

Exercice 4.3. Let $\mathfrak{g}$ be a semisimple Lie algebra. Show that:

1. $\mathfrak{g}^{ss}$ is open and dense in $\mathfrak{g}$,

2. $\mathscr{N}$ is closed, conical,

3. $\mathfrak{g}^{ss}$ and $\mathscr{N}$ are $\mathrm{Aut}(\mathfrak{g})$ stable.

Exercice 4.4.

1. What are the automorphisms of $\mathscr{M}_n(\mathbb{C})$ as an associative algebra?

2. Find an automorphism of the Lie algebra $\mathfrak{gl}_n$ that is not the conjugation by an invertible matrix.

Exercice 4.5.

1. Show that the enveloping algebra of a Lie algebra over a field has no zero-divisors.

2. **a.** Let A be an algebra. Establish a correspondence between certain algebra morphisms

$$A \to \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$$

and derivation of A.

b. Show that for any derivation D of a Lie algebra $\mathfrak{g}$, there exists a unique derivation $\tilde{D} : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g})$ which lifts D. Express $\tilde{D}(x)$ for $x \in \mathfrak{g}$.

3. The enveloping algebra is non-commutative. Nevertheless, show that $\mathbf{U}(\mathfrak{g}) \simeq \mathbf{U}(\mathfrak{g})^{\mathrm{op}}$.

4. Show that the enveloping algebra of a finite dimensional Lie algebra is a noetherian algebra.

5. Find a generalization of the results of 1. and 3..

Exercice 4.6. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. For $x \in \mathfrak{g}$, we let $\mathrm{ad}(x) \in \mathrm{End}(\mathbf{U}(\mathfrak{g}))$ be the endormorphism defined by $\mathrm{ad}(x)(y) = xy - yx$ for any $y \in \mathbf{U}(\mathfrak{g})$.

1. Show that $\mathfrak{g} \subset \mathbf{U}(\mathfrak{g})$ is $\mathrm{ad}(x)$-stable for any $x \in \mathfrak{g}$.

2. Show that $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathbf{U}(\mathfrak{g}))$ is a representation of $\mathfrak{g}$ preserving the filtration of $\mathbf{U}(\mathfrak{g})$.

Exercice 4.7.

1. Recall the Killing form of $\mathfrak{sl}_n$.

2. Compute the Casimir element of $\mathfrak{sl}_2$.

3. What is the Casimir element of $\mathfrak{sl}_n$?

Exercice 4.8. Killing form. Let $\mathfrak{g}$ be a reductive Lie algebra. Make the link between the Killing form of $\mathfrak{g}$ and that of the derived Lie algebra $D(\mathfrak{g})$.

Exercice 4.9. Let $\mathfrak{g}$ be a simple Lie algebra. Show that any $\mathfrak{g}$-invariant bilinear form is proportional to the Killing form.

Exercice 4.10. Basic facts on Poisson Lie/Poisson algebraic groups.