Exercise sheet 3

Thursday, 26 November 2020

Exercice 3.1. The Witt and Virasoro Lie algebras. The Witt algebra is the algebra of polynomial vector fields on $\mathbf{C}^*$ or equivalently the Lie algebra of derivations of $\mathbf{C}[z, z^{-1}]$. Let $L_n = -z^{n+1}\frac{d}{dz}$.

1. Compute the relations between the L_n's. This Lie algebra is denoted Witt.

A central extension of a Lie algebra $\mathfrak{g}$ is an exact sequence

$$0 \to \mathfrak{z} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

such that $[\mathfrak{z}, \hat{\mathfrak{g}}] = 0$. We let

$$C^2(\mathfrak{g}, \mathfrak{z})$$

be the space of linear maps $\beta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{z}$ satisfying the conditions

1. β is antisymmetric,
2. for any $x, y, z \in \mathfrak{g}$,
$$\beta(x, [y, z]) + \beta(y, [z, x]) + \beta(z, [x, y]) = 0.$$

We let $B^1(\mathfrak{g}, \mathfrak{z}) = \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathfrak{z})$ and define $d = - \circ [,] : B^1(\mathfrak{g}, \mathfrak{z}) \to C^2(\mathfrak{g}, \mathfrak{z})$.

2. Show that $H^2(\mathfrak{g}, \mathfrak{z}) := C^2(\mathfrak{g}, \mathfrak{z})/dB^1(\mathfrak{g}, \mathfrak{z})$ classifies central extensions of $\mathfrak{g}$ by $\mathfrak{z}$ up to isomorphism.

3. Show that $H^2(\mathrm{Witt}, \mathbf{C}) = \mathbf{C}$.

The central extension of Witt corresponding to $\beta(L_m \otimes L_n) = \delta_{m,-n}\frac{1}{12}(m^3 - m)$ is called the Virasoro Lie algebra and denoted Vir.

Exercice 3.2. The Heisenberg Lie algebra. The Heisenberg algebra Heis is the Lie algebra with generators $\{a_n : n \in \mathbf{Z}\} \cup \{h\}$ where h is central and for any $m, n \in \mathbf{Z}$,

$$[a_m, a_n] = m\delta_{m,-n}h.$$

Let $h', \mu \in \mathbf{C}$. Show that the following formulas define a representation of Heis on $\mathbf{C}[x_i : i \in \mathbb{N}]$: a_n operates by $\frac{\partial}{\partial x_n}$ is $n > 0$, by multiplication by $-h'nx_{-n}$ is $n < 0$ and by multiplication by μ if $n = 0$ and h operates by multiplication by h'.

1. Show that this action is well-defined.

2. Determine when it is irreducible according to the values of h', μ.

Exercice 3.3. Find an exact sequence

$$0 \to \mathfrak{i} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{i} \to 0$$

such that $\mathfrak{i}$ and $\mathfrak{g}/\mathfrak{i}$ are nilpotent Lie algebras but $\mathfrak{g}$ is not nilpotent.

Exercice 3.4. $\mathfrak{sl}_2$. Let $\mathfrak{sl}_2$ be the sub-Lie algebra of $\mathfrak{gl}_2$ of traceless 2×2 matrices. It is generated by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It can be seen as the free Lie algebra on three generators modulo the relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

1. Show that $\mathfrak{sl}_2$ is a simple Lie algebra.

Let V be a finite dimensional irreducible representation of $\mathfrak{sl}_2$. For $\lambda \in \mathbf{C}$, we let V_λ be the λ-eigenspace of h.

2. Let $v \in V$ be an eigenvector for h with eigenvalue λ. Show that $e \cdot v \in V_{\lambda+2}$ and $f \cdot v \in V_{\lambda-2}$. Deduce that for some $m, n \geq 0$, $e^m \cdot v = 0 = f^n \cdot v$.

3. Take $v \in V \neq 0$ such that $e \cdot v = 0$. For $n \geq 0$, give an explicit formula for $e^n f^n \cdot v$. Deduce that λ is a nonnegative integer.

4. Prove that the trace of h acting on V is 0. Deduce that $\lambda = \dim V - 1$.

5. Classify all the irreducible finite dimensional representations of $\mathfrak{sl}_2$.

Exercice 3.5. Clebsch-Gordan rule. By Exercise 3.4 and the simplicity of $\mathfrak{sl}_2$, the action of h on any finite dimensional representation V of $\mathfrak{sl}_2$ is diagonalizable with integer eigenvalues. For $n \in \mathbf{Z}$, the n-eigenspace of h is denoted $V[n]$. It is called the weight-space (of weight n). The character of V is

$$\mathrm{ch}(V) = \sum_{n \in \mathbf{Z}} \dim V[n] t^n.$$

which can be seen as an element of $\mathbf{Z}[t, t^{-1}]$.

1. Express the compatibility of ch with direct sums and tensor products of representations.

2. Show that two finite dimensional representations of $\mathfrak{sl}_2$ are isomorphic if and only if they have the same character.

3. Let V_m and V_n be respectively the $(m+1)$-st and $(n+1)$-st dimensional irreducible representation of $\mathfrak{sl}_2$. Prove the Clebsh-Gordan rule, that is

$$V_m \otimes V_n \simeq \bigoplus_{i=0}^{\min(m,n)} V_{m+n-2i}$$

as $\mathfrak{sl}_2$-representations.

4. Show that every finite dimensional representation of V appears as a direct summand of a tensor product of the 2-dimensional irreducible representation V_1 of $\mathfrak{sl}_2$.

Exercice 3.6. Polynomial realization of finite dimensional representations of $\mathfrak{sl}_2$.

Let $\mathbf{C}[x, y]$ be the polynomial algebra with two generators. We see it as the algebra of polynomial functions on $\mathbf{C}^2$ and let $\mathfrak{sl}_2$ act on it accordingly (the action of $\mathfrak{sl}_2$ on $\mathbf{C}^2$ is its defining representation).

1. Describe explicitly the action of the generators of $\mathfrak{sl}_2$.

2. What is the subrepresentation given by homogeneous polynomials of some given fixed degree?

Exercice 3.7. We let $\mathfrak{sl}_2$ act on the polynomial ring $\mathbf{C}[u,v]$ by derivations via

$$e \mapsto u\frac{\partial}{\partial v}, \quad f \mapsto v\frac{\partial}{\partial v}, \quad h \mapsto u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

1. Show that this action is well-defined.

2. Show that this infinite dimensional representation is completely reducible and describe its irreducible summands.

Exercice 3.8. Compute the automorphism group of the forgetful functor

$$\mathrm{Rep}(\mathfrak{sl}_2) \to \mathrm{Vect}_{\mathbf{C}}$$

from the category of finite dimensional representations of $\mathfrak{sl}_2$ to the category of finite dimensional complex vector spaces.

Exercice 3.9. $\mathfrak{sl}_3$

1. Give a presentation of the Lie algebra $\mathfrak{sl}_3$ by generators and relations. Hint: consider $e_1 = E_{1,2}, e_2 = E_{2,3}, f_1 = E_{2,1}, f_2 = E_{3,2}, h_1 = E_{1,1} - E_{2,2}$, $h_2 = E_{2,2} - E_{3,3}$.

Let V be an irreducible finite dimensional representation of $\mathfrak{sl}_3$. The presentation by generators and relations given in 1. gives two embeddings $\mathfrak{sl}_2 \to \mathfrak{sl}_3$, namely the subalgebra generated by e_1, f_1, h_1 and the one generated by e_2, f_2, h_2.

For $(k,l) \in \mathbf{Z}^2$, we let $V_{k,l}$ be the intersection of the k-eigenspace of h_1 and the l-eigenspace of h_2. We have

$$V = \bigoplus_{(k,l)\in\mathbf{Z}^2} V_{k,l}.$$

2. Show that $e_1(V_{k,l}) \subset V_{k+2,l-1}$, $e_2(V_{k,l}) \subset V_{k-1,l+2}$, $f_1(V_{k,l}) \subset V_{k-2,l+1}$ and $f_2(V_{k,l}) \subset V_{k+1,l-2}$.

3. Show that any irreducible representation of $\mathfrak{sl}_3$ is generated by a single element (called highest weight vector).

Exercice 3.10. $\mathfrak{sl}_n$**.**

1. Compute the Killing form of $\mathfrak{sl}_n$. In particular, compare it with the trace bilinear form induced from $\mathfrak{gl}_n$.

2. Show that $\mathfrak{sl}_n$ is a simple Lie algebra.

3. Give a presentation by generators and relations of $\mathfrak{sl}_n$ which for $n = 2$ specializes to the presentation given in Exercise 3.4.

Exercice 3.11. Determine the Lie algebra of polynomial vector fields on the projective line $\mathbf{P}^1_{\mathbf{C}}$. Determine the Lie algebra of polynomial global differential operators on $\mathbf{P}^1_{\mathbf{C}}$.