Exercise sheet 1

Thursday, 12 November 2020

Let B be a commutative ring. A B-algebra is the data of a triple (A, m, u) where A is a B-module, $m : A \otimes_B A \to A$ a map of B-modules (the multiplication) satisfying the associativity axiom and $u : B \to A$ a map of B modules (the unit) satisfying the unitality axiom.

A B-coalgebra C is the data of a triple (C, Δ, ϵ) where C is a B-module, $\Delta : C \to C \otimes_B C$ a map of B modules (the comultiplication) satisfying the coassociativity axiom and $\epsilon : C \to B$ a map of B-modules (the counit) satisfying the counitality axiom.

A B-bialgebra is the data of a quintuple $(A, m, u, \Delta, \epsilon)$ where Δ and m are compatible in the sense that Δ and ϵ are morphisms of algebras, or equivalently m and u are morphisms of coalgebras.

A Hopf algebra is the data of a sextuple $(A, m, u, \Delta, \epsilon, S)$ where $(A, m, u, \Delta, \epsilon)$ is a bialgebra and $S : A \to S$ is a B-linear map (the antipode) satisfying the relations

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \epsilon, \quad m \circ (\mathrm{id} \otimes S) \circ \Delta = u \circ \epsilon.$$

Exercice 1.1. Sweedler's notation. Let (C, Δ, ϵ) be a coalgebra. For $c \in C$, there exists elements $c_i, c_i' \in C$ for $i \in I$ (I a finite set) such that

$$\Delta(c) = \sum_{i \in I} c_i \otimes c_i'.$$

Sweedler introduded the notation

$$\Delta(c) = c_{(1)} \otimes c_{(2)}$$

which has to be interpreted as a sum, as above.

1. Express the axioms of coassociativity and counitality using Sweedler's notation.

2. Define a morphism of coalgebras or bialgebras or Hopf algebras and express the properties in terms of Sweedler's notation.

3. Let $(A, m, u, \Delta, \epsilon)$ be a bialgebra. Write the compatibility of m and Δ using Sweedler's notation. Write the condition u has to verify to be a coalgebra morphism in terms of Sweedler's notation.

Exercice 1.2. The antipode. Let C be a B-coalgebra and A a B-algebra. The B-module $\mathrm{Hom}_B(C, A)$ is endowed with the convolution product:

$$f \star g = m_A \circ (f \otimes g) \circ \Delta_C.$$

1. Show that $\star$ is associative.

2. Determine the unit of the convolution algebra $\mathrm{Hom}_B(C, A)$.

3. Let A be a B-bialgebra. Show that an antipode $S : A \to A$ is an inverse to the identity function in the convolution algebra $\mathrm{Hom}_B(A, A)$. As a consequence, the antipode is unique if it exists.

4. Let A be a bialgebra. Explain how it induces a coalgebra structure on $A \otimes_B A$.

5. Show that the antipode is a antihomomorphism $A \to A$, that is for any $a, b \in A$, $S(ab) = S(b)S(a)$.

6. Let H be a Hopf algebra. Show that the following are equivalent:

1. $S^2 = \mathrm{id}$,
2. For any $h \in H$, $S(h_{(2)})h_{(1)} = u \circ \epsilon(h)$,
3. For any $h \in H$, $h_{(2)}S(h_{(1)}) = u \circ \epsilon(h)$.

7. Deduce that $S^2 = \mathrm{id}_H$ if H is commutative or cocommutative.

8. Quantum $\mathfrak{sl}_2$: A non-commutative non-cocommutative Hopf algebra. Let $B = \mathbf{Q}(q)$. We consider the associative B-algebra generated by K, K^{-1}, E, F satisfying the relations

$$KK^{-1} = 1 = K^{-1}K,$$

$$KE = q^2 EK, \quad KF = q^{-1}FK,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

which we denote by $\mathbf{U}_q(\mathfrak{sl}_2)$. It has a comultiplication defined by

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\Delta(K) = K \otimes K,$$

a counit $\epsilon : \mathbf{U}_q(\mathfrak{sl}_2) \to \mathbf{Q}(q)$ defined by

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1,$$

and an antipode defined by

$$S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K) = K^{-1}.$$

Check this defines a genuine Hopf algebra and that $S^2 \neq \mathrm{id}$.

Exercice 1.3. The Hopf algebra of a linear algebraic group. Let k be an algebraically closed field and X an algebraic variety over k (an integral connected scheme of finite type over k). Recall that a regular function $f \in \Gamma(U, \mathcal{O}_X)$ on an open subset $U \subset X$ is characterized by its values on k-points of U (by the Nullstellensatz).

If G is a linear algebraic group ($G = \mathrm{Spec}(k[G])$, say), there are induced operations on $k[G]$:

1. The unit $e : pt = \mathrm{Spec}(k) \to G$ gives a map $u : k[G] \to k$, $f \mapsto f \circ e$,
2. The multiplication m gives a map $\Delta : k[G] \to k[G] \otimes_k k[G] \simeq k[G \times G]$, $f \mapsto f \circ m$,
3. The inverse i gives a map $S : k[G] \to k[G]$, $f \mapsto f \circ i$.

1. Show that $k[G]$ with the natural algebra structure and the operations above is a (commutative) Hopf algebra.

2. Describe explicitly the Hopf algebra of the following linear algebraic groups: the multiplicative group $\mathbf{G}_m$, tha additive group $\mathbf{G}_a$, the general linear group GL_n.

Exercice 1.4. Group schemes. In the same way a linear algebraic group over k is a group object in the category of affine algebraic varieties, one can define a *group object* in any category having products. In particular, if S is a scheme and Sch/S the slice category of schemes over S, a group-scheme over S is a group-object in Sch/S.

1. Assume $S = \mathrm{Spec}(B)$ is an affine scheme. Show that there is an antiequivalence of categories between affine group schemes over S and commutative Hopf algebras over B.

2. Define the notion of representation of a affine group scheme. Translate this definition on terms of Hopf algebras. We obtain the notion of comodule.

Exercice 1.5. Subtleties. **1.** Let G be an algebraic group over k. The group of rational points $G(k)$ is an abstract group but *not* a topological group (for the Zariski topology).

2. Let $n \geq 2$ be an integer. What is the kernel of the morphism of algebraic groups $\mathbf{G}_m \to \mathbf{G}_m$, $t \mapsto t^n$? (It may be useful to define precisely what is meant by “kernel”). What happens if the characteristic of k divides n?

3. **a.** Let G be a group scheme of finite type over k. Show that G is smooth if and only if it is smooth at the neutral element.

b. Let k be a nonperfect field of characteristic $p > 0$ and $t \in k \setminus k^p$ (e.g. $k = \mathbf{F}_p((t))$). Show that the equation $x^{p^2} - tx^p = 0$ defines a subgroup scheme of $\mathbf{G}_a \times \mathbf{G}_a$.

c. Determine G_{red}, the reduced subscheme of G. Show that G_{red} is smooth at the neutral element 0.

d. Show that G_{red} is not smooth. Deduce that G_{red} is not an algebraic group (neither a group scheme) for any map $m : G_{red} \times G_{red} \to G_{red}$.

One can prove that a group scheme of finite type over a field of characteristic zero (non necessarily algebraically closed) is smooth.

Exercice 1.6. Action of a linear group on an affine algebraic variety. Let G be a linear algebraic group acting on an affine algebraic variety X. Show that there exist $n \geq 0$ and closed immersions $G \to \mathrm{GL}_n$, $X \to \mathbf{A}^n$ such that the action of G on X transforms to the natural action of GL_n on $\mathbf{A}^n$.