DUALIZING SHEAF OF A NODAL SINGULARITY

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ABSTRACT. We try to describe as precisely as possible the (shifted) dualizing sheaf of the nodal singularity $\{xy = 0\} \subset \mathbf{C}^2$: as extension of perverse sheaves and we describe its fiber at 0.

Let $X = \{xy = 0\} = A \cup B \subset \mathbf{C}^2$ be the union of the two coordinate axes, where $A = \{y = 0\}$ is the x-axis and $B = \{x = 0\}$ the y-axis. We let $i_A : A \to X$ and $i_B : B \to X$ be the natural inclusions. By abusei of notation, we denote by i_0 the inclusion of $\{0\}$ in either of the spaces X, A and B. We let $j : X \setminus \{0\} \to X$ be the open inclusion of the smooth locus. For a space Y, we let $\underline{\mathbf{Q}}_Y$ denote the constant sheaf on Y. We have an exact sequence of constructible sheaves on X:

$$0 \to \underline{\mathbf{Q}}_X \xrightarrow{a} \underline{\mathbf{Q}}_A \oplus \underline{\mathbf{Q}}_B \xrightarrow{b} \underline{\mathbf{Q}}_{\{0\}} \to 0$$

where $a = (i_A^*, i_B^*)$ and $b = i_0^*$ on the first direct summand and $b = -i_0^*$ on the second. Therefore, we obtain a distinguished triangle in $D^b(X, \mathbf{Q})$:

$$\underline{\mathbf{Q}}_0 \to \underline{\mathbf{Q}}_X[1] \to \underline{\mathbf{Q}}_A[1] \oplus \underline{\mathbf{Q}}_B[1] \to . \tag{0.1}$$

Extremal terms are perverse sheaves on X so $\underline{\mathbf{Q}}_X[1]$ is a perverse sheaf on X. It is not simple.

Proposition 0.1. *The perverse sheaf $\underline{\mathbf{Q}}_X[1]$ is an indecomposable perverse sheaf.*

Proof. Write $\underline{\mathbf{Q}}_X[1] = \mathcal{F} \oplus \mathcal{G}$. Then, $\mathcal{F}$ and $\mathcal{G}$ are perverse sheaves. We have $j^*\underline{\mathbf{Q}}_X[1] = \underline{\mathbf{Q}}_{A\setminus\{0\}}[1] \oplus \underline{\mathbf{Q}}_{B\setminus\{0\}}[1] = j^*\mathcal{F} \oplus j^*\mathcal{G}$. There are essentially two cases: either $j^*\mathcal{G} = 0$ or none of the two perverse sheaves $j^*\mathcal{F}$ and $j^*\mathcal{G}$ vanish. In the first case, $\mathcal{G}$ is supported on $\{0\}$, so $\mathcal{G} = \mathcal{H}^0(\mathcal{G})$ is a direct summand of $\mathcal{H}^0(\underline{\mathbf{Q}}_X[1]) = 0$. In the second case, we have for example $j^*\mathcal{F} = \underline{\mathbf{Q}}_{A\setminus\{0\}}[1]$ and $j^*\mathcal{G} = \underline{\mathbf{Q}}_{B\setminus\{0\}}[1]$. Therefore, $\underline{\mathbf{Q}}_A[1]$ is a simple subquotient of $\mathcal{F}$ and $\underline{\mathbf{Q}}_B[1]$ is a simple subquotient of $\mathcal{G}$. The other subquotients of $\mathcal{F}$ or $\mathcal{G}$ are supported on $\{0\}$. There are no non-zero morphisms $\underline{\mathbf{Q}}_X[1] \to \underline{\mathbf{Q}}_0$, so $\mathcal{F}$ fits into an exact sequence

$$0 \to \underline{\mathbf{Q}}_0^{\oplus m} \to \mathcal{F} \to \underline{\mathbf{Q}}_A[1] \to 0$$

for some $m \geq 0$, and similarly for $\mathcal{G}$:

$$0 \to \underline{\mathbf{Q}}_0^{\oplus m'} \to \mathcal{G} \to \underline{\mathbf{Q}}_B[1] \to 0.$$

Taking the long exact sequence of cohomology and the fact that $\mathcal{H}^0(\mathcal{F}) = 0$, we get the exact sequence

$$0 \to \mathcal{H}^{-1}(\mathcal{F}) = \mathcal{F}[-1] \to \underline{\mathbf{Q}}_A \to \underline{\mathbf{Q}}_0^{\oplus m} \to 0,$$

and the similar exact sequence for $\mathcal{G}$:

$$0 \to \mathcal{H}^{-1}(\mathcal{G}) = \mathcal{G}[-1] \to \underline{\mathbf{Q}}_B \to \underline{\mathbf{Q}}_0^{\oplus m'} \to 0.$$

We deduce that $0 \leq m, m' \leq 1$ and by taking the fiber over the origin, $m + m' = 1$. If for example $m = 0$, we have $\mathcal{F} = \underline{\mathbf{Q}}_A[1]$ and therefore we would have a non-zero morphism

$$\underline{\mathbf{Q}}_A \to \underline{\mathbf{Q}}_X,$$

but such morphisms do not exist. Indeed, given such a morphism, we have the commutative diagram:

$$\begin{array}{ccc} \underline{\mathbf{Q}}_A(X) = \mathbf{Q} & \longrightarrow & \underline{\mathbf{Q}}_X(X) = \mathbf{Q} \\ \downarrow & & \downarrow \\ \underline{\mathbf{Q}}_A(X \setminus A) = 0 & \longrightarrow & \underline{\mathbf{Q}}_X(X \setminus A) = \mathbf{Q} \end{array}$$

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and the right-most vertical arrow is an isomorphism. This leads to a contradiction and therefore, $\underline{\mathbf{Q}}_X[1]$ is indecomposable. □

Proposition 0.2. *The complex* $\mathbb{D}(\mathbf{Q}_X[1])$ *is an indecomposable perverse sheaf on* X*. Its restriction to the complement of* $\{0\}$ *is isomorphic to* $\mathbf{Q}_{X\setminus\{0\}}[1]$ *and its fiber over* 0 *is isomorphic to the complex of vector spaces* $\simeq (\mathbf{Q}^{\oplus 2}[1]) \oplus \mathbf{Q}$.

Proof. Since $\mathbb{D}$ is an anti-autoequivalence of the category of perverse sheaves, the dual of an indecomposable perverse sheaf is again an indecomposable perverse sheaf. Then, taking the dual of the exact triangle 0.1, we obtain an exact triangle

$$\underline{\mathbf{Q}}_A[1] \oplus \underline{\mathbf{Q}}_B[1] \to \mathbb{D}(\underline{\mathbf{Q}}_X[1]) \to \underline{\mathbf{Q}}_0 \to .$$

Letting $\mathscr{F} = \mathbb{D}(\underline{\mathbf{Q}}_X[1])$, the long exact sequence of cohomology induced can be written:

$$0 \to \underline{\mathbf{Q}}_A \oplus \underline{\mathbf{Q}}_B \to \mathcal{H}^{-1}(\mathscr{F}) \to 0 \to 0 \to \mathcal{H}^0(\mathscr{F}) \to \underline{\mathbf{Q}}_0 \to 0.$$

Taking the fiber over 0, we obtain the exact sequence of vector spaces:

$$0 \to \underline{\mathbf{Q}} \oplus \underline{\mathbf{Q}} \to H^{-1}(\mathscr{F}_0) \to 0 \to 0 \to H^0(\mathscr{F}_0) \to \mathbf{Q} \to 0$$

from which we deduce that $\mathscr{F}_0 \simeq (\mathbf{Q}^{\oplus 2}[1]) \oplus \mathbf{Q}$. □

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