Comptage des représentations des algèbres lisses sur les corps finis (avec Fabian Korthauer)

Motivations: * Understand enumerative invariants of 1, 2, 3 dimensional categories

* Connect geometric representation theory, enumerative geometry and geometric group theory.

* Provide new examples of well-behaved categories to play with and define new moduli spaces.

Introduction

$R \subset \mathbb{C}$ finitely generated ring.

e.g. $R = \mathbb{Z}$

$R = \mathbb{Z}[\xi_p]$ $\quad \xi_p$ p-th root of unity (primitive)

$\simeq \mathbb{Z}[x]/(\phi_p)$ — p-th cyclotomic polynomial, $\phi_p(x) = \frac{x^p - 1}{x - 1}$

$\mathbb{F}_q$ finite field with q elements

$\subset \overline{\mathbb{F}_q}$ algebraic closure.

$$R \otimes_{\mathbb{Z}} \overline{\mathbb{F}_q} \simeq \begin{cases} \overline{\mathbb{F}_q}[x]/x^p & \text{if } p \mid q \\ \overline{\mathbb{F}_q}^{\,p} & \text{if } p \nmid q \end{cases} \qquad \begin{bmatrix} \text{not smooth} \\ \\ \text{smooth.} \end{bmatrix}$$

R a ring.

A a finitely presented R-algebra: finitely many gens, eqns.

$$A = R\langle x_1, \dots, x_n\rangle \big/ \langle f_1(x_i), \dots, f_m(x_i)\rangle$$

Sometimes: $R = \mathbb{C}$ or $R = \mathbb{F}_q$ finite field

We are interested in the case when "A is smooth over R" (defined later)

A is usually a noncommutative algebra!

ex. of finitely presented algebras (over R)

1) $A = R[x_1, \dots, x_n] = R\langle x_1, \dots, x_n\rangle \big/ \langle\langle [x_i, x_j] = 0 \mid 1 \leq i, j \leq n \rangle\rangle$.

polynomial algebra

(2) $A = RQ$ path algebra of a quiver $Q = (\underbrace{Q_0}_{\text{vertices}}, \underbrace{Q_1}_{\text{arrows.}})$

= vector space generated by paths in Q
multiplication is the concatenation of path

$RQ_0 = R^{Q_0}$ product algebra, $e_i \in RQ_0$ idempotents

$$A = R^{Q_0}\langle \alpha : \alpha \in Q_1 \rangle \Big/ \left\langle\left\langle \begin{array}{l} e_i \alpha e_j = \delta_{i s(\alpha)} \delta_{j t(\alpha)} \alpha \\ \alpha\beta = \delta_{t(\alpha) s(\beta)} \alpha\beta. \end{array} \right\rangle\right\rangle$$

(3) $G = \langle g_1, \dots, g_n \,/\, r_1(g_1, \dots, g_n), \dots, r_m(g_1, \dots, g_n) = 1 \rangle$ finitely presented group

$$\underset{\text{group algebra}}{R[G]} = R\langle x_1, \dots, x_n, y_1, \dots, y_m \rangle \Big/ \left\langle\left\langle \begin{array}{l} x_i y_i = y_i x_i = 1 \\ r_j(x_1, \dots, x_n) = 1 \end{array} \right\rangle\right\rangle$$

(4) Most of the algebras you can think of.

(5) $End(P)$, P projective representation of a smooth algebra

Representation spaces of A [can think of $R = \mathbb{C}$ for more concreteness]

$d \in \mathbb{N}$ dimension

- $\mathrm{Rep}(d, A) = \mathrm{Hom}_{R\text{-alg}}(A, \mathrm{Mat}_{d\times d}(R))$

$$= \left\{ M_1, \dots, M_n \in \mathrm{Mat}_{d\times d}(R) \;\middle|\; f_1(M_1, \dots, M_n) = \dots = f_m(M_1, \dots, M_n) = 0 \right\}$$

Does not depend on the presentation of A:

choice of generators x_i and relations f_j.

If

$$A \cong \underset{\uparrow \varphi}{R\langle x_1, \dots, x_n\rangle} \Big/ \underbrace{\langle f_1, \dots, f_m\rangle}_{I_f}$$

$$\cong R\langle y_1, \dots, y_s\rangle \Big/ \underbrace{\langle g_1, \dots, g_t\rangle}_{I_g}$$

$\varphi(y_i) = h_j(x_1, \dots, x_n)$; can use the h_j's to define an isomorphism between the rep. spaces.

- $GL_d(R)$ - action on $\mathrm{Rep}(d, A)$ by simultaneous conjugation.

$GL_d(R)$ - orbits correspond to isomorphism classes of A-representations.

- $\mathcal{M}_{A,d} := [\mathrm{Rep}(A,d) / GL_d]$ quotient stack

stack of d-dimensional reps over A.

It is an R stack.

- $\mathcal{M}_A := \bigsqcup_{d \in \mathbb{N}} \mathcal{M}_{A,d}$.

What does it look like?

1. $A = \mathbb{C}[x_1, \dots, x_n]$

$$\mathrm{Rep}(A,d) = \{ (M_1, \dots, M_n) \in \mathrm{Mat}_{d\times d}(\mathbb{C}) \mid [M_i, M_j] = 0 \}$$

$$\cong \begin{cases} \mathbb{C}^n & \text{if } d = 1 \\ \mathrm{Mat}_{d\times d}(\mathbb{C}) & \text{if } n = 1 \\ \text{singular instead!} & \\ n=2: \text{ Commuting variety of } \mathfrak{gl}_n. \text{ Very intricate geometry.} & \end{cases}$$

$\mathcal{M}_{A,d}$ = stack of torsion sheaves of length d on the affine space $\mathbb{A}^n_{\mathbb{C}}$.

2. $\underline{d} \in \mathbb{N}^{Q_0}$, $d = \sum_{i \in Q_0} d_i$.

$$\mathrm{Rep}(\mathbb{C}Q, \underline{d}) = \prod_{\alpha \in Q_1} \mathrm{Hom}(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}}).$$

(3) $\mathrm{Rep}(\mathbb{C}[G], d) =$ space of d-dimensional representations of d

$\circlearrowleft GL_d$

GL_d-orbits correspond to isomorphism classes of G-representations

ex. G finite. The rep. theory of G is semisimple and therefore $\mathrm{Rep}(\mathbb{C}[G], d)$ has finitely many GL_d-orbits. Actually, each GL_d-orbit corresponds to a connected component of $\mathrm{Rep}(\mathbb{C}[G], d)$.

(4) A any algebra

$\mathfrak{M}_{A,d}$ = stack of length d-torsion sheaves on $\mathrm{Spec}(A)$.

Of course, $\mathrm{Spec}(A)$ only makes sense when A is commutative.

Two aspects: (1) counting representations : work over finite fields $\mathbb{F}_q$

(2) doing geometry : work over the field of complex numbers $\mathbb{C}$.

Smooth algebras [Cuntz - Quillen]

A an R-algebra (possibly non-commutative)

The smoothness of A is an algebraic property ensuring that $\forall d \in \mathbb{N}$, $\mathrm{Rep}(A,d)$ is a smooth scheme over $\mathrm{Spec}\, R$

$$0 \to \underbrace{\Omega^1_{A/R}} \to A \otimes_R A \xrightarrow{m} A \to 0$$

asked to be projective as an $A \otimes_R A^{op}$ - module.

equivalently, it has the lifting property for nilpotent extensions:

$\forall$ R-algebra B, $I \subset B$ 2-sided nilpotent ideal ($I^n = 0$ for $n \gg 0$)

and any $A \to B/I$,

$$\begin{tikzcd} A \arrow[r, "\exists \tilde{f}"] \arrow[dr] & B \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ & B/I \end{tikzcd}$$

① $R[x_1, \ldots, x_n]$ not smooth if $n \geq 2$

② smooth!

③ smooth for some groups $\to$ virtually free groups.

④ A commutative. $\mathcal{M}_{A,d}$ commutative ($\forall d$) if and only if $\mathrm{Spec}\, A$ is a smooth affine curve!

Counting functions

A $\mathbb{F}_q$- algebra

$d \in \mathbb{N}$

$M_{A,d}(q^n) = \#\{$ isoclasses of d-dimensional reps of A over $\mathbb{F}_q \}$

$I_{A,d}(q^n) = \#\{$ isoclasses of indecomposable d-dimensional reps of A over $\mathbb{F}_q \}$

$A_{A,d}(q^n) = \#\{$ isoclasses of absolutely indecomposable d-dimensional reps of A over $\mathbb{F}_q \}$

The main result

Theorem (H-Korthauer).

① Let A be a smooth algebra over $\mathbb{F}_q$.
If $\mathrm{Rep}(A,d)$ has a polynomial number of points over $\mathbb{F}_{q^n}$ ($\forall d$), then the functions

$$M_{A,d}(q^n),\ I_{A,d}(q^n),\ A_{A,d}(q^n) \text{ are polynomials}$$

in q^n.

finitely gen. subring.

② Let A be a smooth algebra over $R \subset \mathbb{C}$.

* If A has strongly polynomial count rep. spaces, the polynomials $M_{A,d}$, $I_{A,d}$ and $A_{A,d}$ can be identified with E-polynomials of some mixed Hodge structures on some (singular) algebraic varieties.

* If $\mathcal{M}_{A_{\mathbb{C}},d}$ is pure $\forall d \in \mathbb{N}$, then the polynomials $M_{A,d}$ and $A_{A,d}$ have nonnegative coefficients ($\forall d \in \mathbb{N}$).

Refinement : We can refine the counting functions as follows.

In case ① Theorem, we let $\mathrm{Rep}(A) := \coprod_{d \in \mathbb{N}} \mathrm{Rep}(A, d)$ and

$\mathcal{T}_A :=$ monoid of connected components of $\mathrm{Rep}(A)$

Of course, we have a surjective morphism $\mathcal{T}_A \longrightarrow \mathbb{N}$.

In case ② theorem, we $\mathcal{T}_A$ is the monoid of connected components of $\mathrm{Rep}(A)$ as an R-scheme

Remark : For any $R \to R'$ morphism of rings, $A' = A \otimes_R R'$, there is a canonical morphism of monoids $\mathcal{T}_{A'} \to \mathcal{T}_A$ since $\mathrm{Rep}(A', d) = \mathrm{Rep}(A, d) \underset{\mathrm{Spec}\, R}{\times} \mathrm{Spec}\, R' \longrightarrow \mathrm{Rep}(A, d)$ and so a connected component of $\mathrm{Rep}(A', d)$ is sent in a component of $\mathrm{Rep}(A, d)$

In particular, if A is an R-algebra, if some connected components of $\mathrm{Rep}(A)$ is not geometrically connected, $\mathcal{T}_{A \otimes_R \overline{\mathbb{F}_q}}$ or $\mathcal{T}_{A \otimes_R \mathbb{C}}$ may have strictly more connected components than $\mathcal{T}_0$

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* This is usually a strict refinement :

quivers : $Q = (Q_0, Q_1)$ $\quad \pi_0(\mathrm{Rep}(\mathbb{C}Q, d)) \simeq \left\{ (d_i)_{i \in Q_0} \,\middle|\, \sum_{i \in Q_0} d_i = d \right\}$.

* G finite group $\quad \pi_0(\mathrm{Rep}\, \mathbb{C}[G]) \simeq \mathbb{N}^S$

$S = \{\text{simple reps of } G\}/\sim$

* Groups $G = \mathbb{Z}/2 * \mathbb{Z}/3$ free product $\quad \pi_0(\mathrm{Rep}\, G) \cong \{(a,b,\alpha,\beta,\gamma) \in \mathbb{N}^5 \mid a+b = \alpha+\beta+\gamma\}$

Theorem has 2 statements

① Polynomiality → relating the number of reps of A with the stacky # of points of $\mathfrak{M}_{A,d}$.

② Positivity statement → develop cohomological DT-theory for 3CY completions of smooth algebras.

Case previously known (and well-studied): quivers.

① Kac (quivers), Schiffmann (curves) [not in our framework but we adapt his method]

② a) Hausel - Letellier - Rodriguez-Villegas

b) Davison.

For positivity, we adapt Davison's method.

① <u>Polynomiality</u> A a $\mathbb{F}_q$-algebra.

* Counting orbits is not a very geometric operation: it is counting points of the set-theoretic quotient

$$\operatorname{Rep}(A,d)/GL_d \,.$$

* inertia spaces / stacks:

$$I\operatorname{Rep}(A,d) = \left\{ (M_1,\dots,M_n,f) \in \operatorname{Rep}(A,d)\times GL_d \;\middle|\; M_i f = f M_i \;\forall i \right\} \circlearrowleft GL_d$$

simultaneous conjugation

Quotient stack $\mathcal{M}^{I}_{A,d} = \left[I\operatorname{Rep}(A,d)/GL_d \right]$

The stacky number of $\mathbb{F}_{q^n}$-points of $\mathcal{M}^{I}_{A,d}$ is exactly $M_{A,d}(q^n)$ [Burnside formula].

$$= \sum_{(\underline{M},f)\in \underbrace{I\operatorname{Rep}(A,d)(\mathbb{F}_{q^n})}_{\mathbb{F}_{q^n}\text{-points}}} \frac{1}{\#\operatorname{Stab}_{GL_d}(M,f)} =: \operatorname{vol}\left(I\operatorname{Rep}(A,d)(\mathbb{F}_{q^n})\right)$$

Want to prove the "polynomiality" of this volume.

↳ rational fraction in q^n.

* nilpotent inertia stack
& endomorphism

$$I^{nil}\mathrm{Rep}(A,d) = \{(\underline{M}, f) \mid f \text{ nilpotent}\}$$
$$I^{End}\mathrm{Rep}(A,d) \{(\underline{M}, f) \mid f \text{ arbitrary}\}$$

* Polynomiality of the volumes of $I^{*}\mathrm{Rep}(A,d)(\mathbb{F}_q)$ for $* \in \{\ , nil, End\}$ are equivalent: choose the most convenient.
* $I^{nil}\mathrm{Rep}(A,d)$ has the Jordan stratification w.r.t. f:

For $\underline{x}=(x,f) \in I^{nil}\mathrm{Rep}(A,d)$, we define the Jordan type of $\underline{x}$ as a tuple $(\alpha_1 \dots \alpha_r) \in \mathcal{T}_A^{r}$ such that

$$[x] = \sum_{i=1}^{r} i\alpha_i, \qquad \text{in the following way.}$$

$\cap$

$\mathcal{T}_A$, connected component of x

x corresponds to a representation M of A and f is an endomorphism of M.

We have a chain of A-representations

$$\mathrm{im} f^0 = M \underset{f}{\supsetneq} \mathrm{im} f \underset{f}{\supsetneq} \mathrm{im} f^2 \dots \underset{f}{\supsetneq} \mathrm{im} f^s \supsetneq 0$$

where s is the nilpotency index of f.

$$\mathrm{im} f^{r-1}/\mathrm{im} f^{r} \twoheadrightarrow \mathrm{im} f^{r}/\mathrm{im} f^{r+1}$$

$$\alpha_r = \left[\ker \operatorname{im} f^{r-1}/\operatorname{im} f^r \to \operatorname{im} f^r/\operatorname{im} f^{r+1}\right].$$

We let $\underline{\alpha} = (\alpha_1, \dots, \alpha_s) =: J(x, f)$

Jordan stratification of the nilpotent inertia stack/space

For $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ s.t. $\sum i\alpha_i = \alpha$, we let

$$I^{\underline{\alpha}}\operatorname{Rep}(A, \alpha) = \left\{(x, f) \in I\operatorname{Rep}^{nil}(A, \alpha) \;\middle|\; J(x, f) = \underline{\alpha}\right\}$$

It is a locally closed subspace of $I\operatorname{Rep}^{nil}(A, \alpha)$

We define the moduli stack of flags of A-representations

$$\mathcal{F}_{\underline{\alpha}} = \left\{F_1 \twoheadrightarrow F_2 \twoheadrightarrow \dots \twoheadrightarrow F_s \;\middle|\; [F_i] = \alpha_i + \dots + \alpha_s = f_i(\underline{\alpha})\right\}$$

At the stack level, we have maps:

$d = |\alpha|$ total dimension.

$$I^{\underline{\alpha}}\operatorname{Rep}(A, \alpha)/GL_d \xrightarrow{\varphi} \mathcal{F}_{\underline{\alpha}}$$

$$(x, f) = (M, f) \longmapsto \left(\operatorname{im} f^0/\operatorname{im} f^1 \to \operatorname{im} f^1/\operatorname{im} f^2 \to \dots \to \operatorname{im} f^{s-1}/\operatorname{im} f^s\right)$$

and

$$\mathcal{F}_{\underline{\alpha}} \xrightarrow{\psi} \prod_{i=1}^{s} \mathcal{M}_{A, \alpha_i}$$

$$(F_1 \xrightarrow{g_1} \dots \xrightarrow{g_{s-1}} F_s \xrightarrow{g_s} 0) \longmapsto (\ker g_1, \dots, \ker g_s)$$

<u>Fact</u>: Fibers of φ, ψ are affine spaces. (with a small lie)

- $\mathrm{vol}(\text{fiber of } \varphi) = \prod_{k \geq 0} q^{-\chi(f_k(\underline{\alpha}), f_{k+1}(\underline{\alpha}))}$

  $$f_k(\underline{\alpha}) = \sum_{j \geq k} \alpha_j .$$

- $\mathrm{vol}(\text{fiber of } \psi) = - \sum_{j > k} q^{\chi(\alpha_j, \alpha_k)}$

where

$$\chi : \pi_0(\mathrm{Rep}(A)) \times \pi_0(\mathrm{Rep}(A)) \longrightarrow \mathbb{Z}$$

is the Euler form of A :

$\forall$ M,N rep. of A,

$$\chi([M],[N]) = \mathrm{hom}(M,N) - \mathrm{ext}^1(M,N).$$

Implicitly, we use the fact that the quantity on the right only depends on $[M]$ and $[N]$. This comes from the fact that there is a 2-term complex of vector bundles on $\mathrm{Rep}(A) \times \mathrm{Rep}(A)$ such that the cohomology of the fiber over (x,y) computes $\mathrm{Hom}(M,N)$ and $\mathrm{Ext}^1(M,N)$.

So $\mathrm{vol}\ I^{\alpha}\mathrm{Rep}(A,\alpha)[\mathbb{F}_q] = \left(\prod_{j=1}^{s} \mathrm{vol}\,\mathrm{Rep}(A,\alpha_j) \right) q^{*}$

② Positivity A an R-algebra, $R \subset \mathbb{C}$ finitely generated subring s.t. $\mathrm{Rep}(A,d)$ is strongly polynomial count

- Develop the machinery of cohomological Donaldson-Thomas theory for the algebras A, T^*A w/ canonical non-commutative moment map and $T^*A\langle x\rangle$ with its canonical potential.

- Upshot : Find a mixed Hodge structure on $\mathrm{Rep}(T^*A, \alpha)/\!/GL_d$ $\forall \alpha \in \pi_0(\mathrm{Rep}\,A)$ whose Hodge polynomial give are exactly the Kac polynomials $A_{A,\alpha}(q)$ of A.

  $\underbrace{\text{structure on}}_{BPS_{A,\alpha}}$

  * classical story for quivers : Davison
  * this generalization introduces new ingredients in the proof, e.g. Kac formula (for quivers) is replaced by the use of Lefschetz fixed point formula for the Frobenius endomorphism.