BASE-CHANGE MAP AND DUALIZING SHEAF

LUCIEN HENNECART

Abstract. In this short note, we investigate the interaction between base-change and dualizing sheaf.

We consider the Cartesian square of finite-type separated complex schemes

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & T \end{array} .$$

Proposition 0.1. *There is a natural base-change transformation $h^*f^! \to k^!g^*$.*

Proof. We precompose with g^* and postcompose with h^* the natural isomorphism $f^!g_* \cong h_*k^!$ to obtain the isomorphism $h^*f^!g_*g^* \cong h^*h_*k^!g^*$. There is a unit morphism $\mathrm{id} \to g_*g^*$ which provides us with a morphism $h^*f^! \to h^*f^!g_*g^*$ and a counit morphism $h^*h_* \to \mathrm{id}$ giving in turn $h^*h_*k^!g^* \to k^!g^*$. We obtain the natural transformation $h^*f^! \to k^!g^*$ by composing the three morphisms. □

Proposition 0.2. *The natural transformation of Proposition 0.1 is an isomorphism if f is smooth.*

Proof. If f is smooth, then k is also smooth and if d denotes the common relative dimension of f and k, we have $f^! = f^*[2d]$ and $k^! = k^*[2d]$. The proposition then follows from $(f \circ h)^* \cong h^*f^*$ and $(g \circ k)^* \cong k^*g^*$. □

In general, the natural transformation of Proposition 0.1 is not an isomorphism. We give a counterexample in which f is l.c.i. (i.e. quasi-smooth in the derived sense).

Namely, we let $T = \mathbf{C}$ with coordinate λ and $Y = \{(x, y, \lambda) \in \mathbf{C}^3 \mid xy = \lambda\} \subset \mathbb{C}^3$ with coordinates (x, y, λ). The morphism f sends (x, y, λ) to λ and we let $Z = \{0\}$, $X = \{xy = 0\} \subset \mathbf{C}^2$.

Lemma 0.3. *The algebraic variety Y is smooth.*

Proof. The variety Y is the zero locus of the function $\phi\colon \mathbf{C}^3 \to \mathbf{C}$, $(x, y, \lambda) \mapsto xy - \lambda$. The differential of ϕ at (x, y, λ) is given by the vector $(y, x, -1)$ and so is surjective at any point. Therefore, Y is smooth. □

Corollary 0.4. *The morphism $f\colon Y \to \mathbf{C}$, $(x, y, \lambda) \mapsto \lambda$ is l.c.i.*

Proof. The map $f\colon Y \to \mathbf{C}$ can be factored as the regular immersion $f_1\colon Y \to \mathbf{C}^3$ (immersion of a smooth subvariety) followed by the smooth map $f_2\colon \mathbf{C}^3 \to \mathbf{C}$, $(x, y, \lambda) \mapsto \lambda$. Therefore, it is l.c.i. □

Proposition 0.5. *We have $f^!\mathbf{Q}_T \cong \mathbf{Q}_Y[2]$.*

Proof. We have $f^!\mathbf{Q}_T \cong \mathbb{D}f^*\mathbb{D}\mathbf{Q}_T$ and since $T = \mathbf{C}$ is smooth of dimension 1, $\mathbb{D}\mathbf{Q}_T \cong \mathbf{Q}_T[2]$. Therefore, $f^!\mathbf{Q}_T \cong \mathbb{D}(\mathbf{Q}_Y[2])$ and since $\mathbf{Y}$ is smooth of dimension 2, $\mathbb{D}\mathbf{Q}_Y \cong \mathbf{Q}_Y[4]$. We conclude that $f^!\mathbf{Q}_T \cong \mathbf{Q}_Y[2]$. □

We let $Z = \{0\} \subset T$.

Date: September 19, 2024.

Proposition 0.6. *We have*

(1) $h^*f^!\mathbf{Q}_T \cong \mathbf{Q}_X[2]$

(2) $k^!g^*\mathbf{Q}_T \cong \mathbb{D}\mathbf{Q}_X$.

In particular, the natural transformation $h^*f^!\mathbf{Q}_T \to k^!g^*\mathbf{Q}_T$ of Proposition 0.1 is not an isomorphism.

Proof. The point (2) is immediate: $k^!g^*\mathbf{Q}_T \cong k^!\mathbf{Q}_{\{0\}} \cong \mathbb{D}\mathbf{Q}_X$.

The point (1) follows directly from Proposition 0.5. □

Remark 0.7. We can describe the natural morphism $h^*f^!\mathbf{Q}_T \cong \mathbf{Q}_X[2] \to k^!g^*\mathbf{Q}_T \cong \mathbb{D}\mathbf{Q}_X$ as follows. We have non-split distinguished triangles (see [Hen22]):

$$\mathbf{Q}_0[1] \to \mathbf{Q}_X[2] \xrightarrow{a} \mathbf{Q}_{\{x=0\}}[2] \oplus \mathbf{Q}_{\{y=0\}}[2] \to$$

and its shifted dual

$$\mathbf{Q}_{\{x=0\}}[2] \oplus \mathbf{Q}_{\{y=0\}}[2] \xrightarrow{b} \mathbb{D}\mathbf{Q}_X \to \mathbf{Q}_0[1] \to .$$

Then, the natural morphism $h^*f^!\mathbf{Q}_T \to k^!g^*\mathbf{Q}_T$ is $b \circ a$.

We shall now give a sufficient condition (some kind of equisingularity condition) that ensures that the morphism of Proposition 0.1 is an isomorphism.

Proposition 0.8. *We assume that for any $y \in Y$, there is an analytic open neighbourhood V of y that is isomorphic to $U \times (V \cap f^{-1}(f(y)))$ over T for some analytic open neighbourhood $U \subset T$ of $f(y)$. Then, the morphism of Proposition 0.1 is an isomorphism.*

Proof. The assumption of the proposition means that we have a diagram of analytic spaces with Cartesian squares

$$\begin{array}{ccc} g^{-1}(U) \times V_y & \xrightarrow{m} & U \times V_y \\ \downarrow{\scriptstyle v} & \lrcorner & \downarrow{\scriptstyle u} \\ X & \xrightarrow{h} & Y \\ \downarrow{\scriptstyle k} & \lrcorner & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & T \end{array} \quad , \tag{1}$$

where we define $V_y = U \times (V \cap f^{-1}(f(y)))$. It suffices to prove that v^* applied to the morphism $h^*f^! \to k^!g^*$ is an isomorphism. This gives exactly the morphism of Proposition 0.1 for the outer Cartesian square of (1). It is then straightforward to check that $m^*(f \circ u)^!\mathbf{Q}_T \cong \mathbf{Q}_{g^{-1}(U)} \boxtimes \mathbb{D}\mathbf{Q}_{V_y} \cong g^*(k \circ v)^!\mathbf{Q}_T$. □

Remark 0.9. In Proposition 0.8, we may consider smooth neighbourhoods V and U of y and $f(y)$ respectively instead of analytic open neighbourhoods. The key point is that surjective smooth pullbacks are fully faithful.

References

[Hen22] Lucien Hennecart. *Dualizing Sheaf of a Nodal Singularity*. https://www.maths.ed.ac.uk/~lhenneca/dualizing-sheaf-node.pdf. Accessed: 2024-09-19. Mar. 2022.

School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Edinburgh EH9 3FD

Email address: lucien.hennecart@ed.ac.uk