Mirror Symmetry of 3d $\mathcal{N} = 4$ Abelian Gauge Theories (joint with Andrew Ballin, Thomas Creutzig, Tudor Dimofte)

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Outline

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- The category of line operators should be a braided tensor category (BTC).
- The work of Hilburn-Raskin gives chiral categories, but they are too big to possess braided tensor structure.

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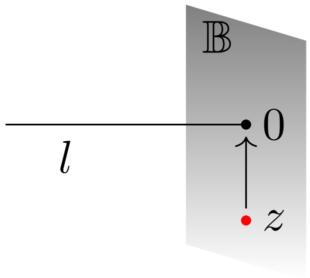

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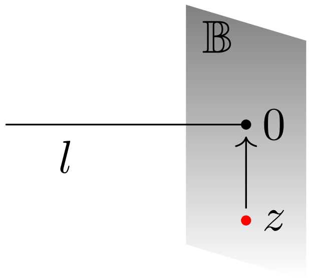

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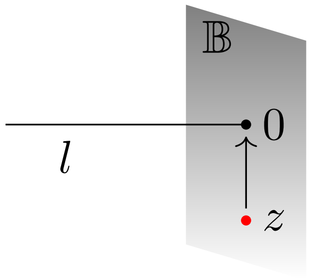

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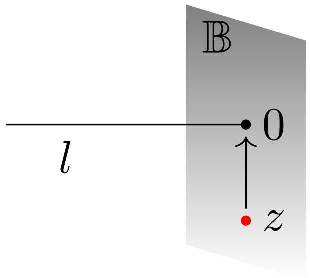

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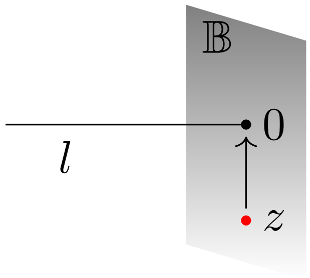

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- For $T_{A,q}$, $\mathbb{B}$ is the Neumann boundary condition, and for $T_{B,q}$, $\mathbb{B}$ is the Dirichlet boundary condition.
- Method was used in the previous work with A. Ballin.

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- This has level $-q^T q$:

$$J^i(z)J^j(w) \sim \frac{-\sum_a q_{ia}q_{ja}}{(z-w)^2}$$

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- The BRST differential

$$\tilde{Q}_{BRST} = \sum_i \oint \mathrm{d}z c^i \tilde{J}^i$$

Satisfies $\tilde{Q}^2 = 0$.

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Definition

The boundary VOA $V_{A,q}$ for the Neumann boundary condition of $T_{A,q}$ is defined to be the cohomology:

$$V_{A,q} := H^* \left(V_{\beta\gamma}^{\otimes n} \otimes V_{bc}^{\otimes n} \otimes V_{bc}^{\otimes r}, \tilde{Q} \right)$$

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$$N^i(z)|m\rangle \sim \frac{(q^T qm)_i/2}{z}, \ E^i(z)|m\rangle \sim \frac{m_i}{z}$$

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- We denote these simple modules by $V_{\text{per},m}$.

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Definition (and claim)

We claim that $V_{per,m}$ *are monopole operators, and define the boundary VOA for Dirichlet b.c. of* $T_{B,q}$ *to be:*

$$V_{B,q} := \bigoplus_m V_{per,m}$$

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- We will assume that q induces the following short exact sequence:

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- Expectation: Neumann on A corresponds to Dirichlet on B (Bullimore-Dimofte-Gaiotto-Hilburn).

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Theorem (A. Ballin, T. Creutzig, T. Dimofte, W. N.)

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Remark. This is obtained by using a free field realization of VOAs on both sides. For the left hand side, BRST cohomology of Fock modules of the Heisenberg VOA is well known.

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- Category of line operators: modules of $V_{A,q}$ or $V_{B,q}$.
- The problem: one needs to be careful when applying Huang-Lepowsky-Zhang to get braided tensor category (BTC).
- We can use the idea of simple current extensions (Creutzig-McRae-Yang, Creutzig-Kanade-Linshaw).

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- $KL_q^{[0]}$, the sub category of modules having trivial monodromy with $V_{B,q}$.

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Definition

We define the category of line operators for $T_{B,q}$, denoted by $\mathcal{C}_{B,q}$, to be $KL_q^{[0]}/\mathbb{Z}^r$.

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- Problems:
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- Another approach: $V_{A,q}$ is Morita equivalent to an extension of $V_{\beta\gamma}^{\otimes n}$.

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- $V_{\beta\gamma}^{\otimes n}$ has $\mathbb{Z}^n$ lattice of simple currents, given by:

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Proposition

There is an isomorphism of VOAs:

$$V_{ext} \otimes V_{bc}^{\otimes n} \cong V_{A,q} \otimes W$$

Here W and $V_{bc}^{\otimes n}$ have trivial category of modules.

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$$M \leftrightarrow M \times \sigma^{qm} V_{\beta\gamma}^{\otimes n}.$$

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Remark. The proof is based on the relation between $V_{\beta\gamma}$ and $V(\widehat{\mathfrak{gl}}(1|1))$ (w A. Ballin). All these come from subquotient of Kazhdan-Lusztig category of $V(\widehat{\mathfrak{gl}}(1|1))$.

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- The quotient by $\mathbb{Z}^r$ is identified with the action of the spectral flow σ^{qm}.
- Recall that $\mathcal{K}^* = (1 + z\mathcal{O}) \times \mathbb{C}^* \times \mathbb{Z}$. Strongly equivariant D modules on $\mathcal{K}^n$.

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- What is the boundary VOA for the non-abelian gauge theories, and what is the category of line operators.

Last word

Thank you!