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Cohomological integrality for symmetric representations of reductive groups

* Donaldson-Thomas theory ⇝ enumerative geometry of compact CY 3-folds (DT, end of 1990s)

  $\mathcal{M}$ moduli space of sheaves on X 3CY

  $[\mathcal{M}]^{\mathrm{vir}}$ virtual fundamental class, coh degree 0

  ⇝ en. invariants by taking the degree

[
* extended to enumerative geometry of 3CY categories

3CY categories : Coh (compact 3CY manifold)

Coh$_c$ (3CY)

Rep (Jac (Q, W))

[dg-Jacobi algebra of quiver Q with potential W.
]

* DT invariants are Euler characteristics (Behrend, 2000s)

  $\nu : \mathcal{M} \longrightarrow \mathbb{Z}$ constructible function (Behrend function)

$$DT = \int_{\mathcal{M}} \nu \, d\chi = \sum_{a \in \mathbb{Z}} \chi(\nu^{-1}(a))\, a .$$

lot of authors now

* Cohomological DT-theory $\rightsquigarrow$ refined invariants

$\exists\ \mathcal{DT} \in \mathrm{Perv}(\mathcal{M})$ such that

$$\nu(x) = \chi(\mathcal{DT}_x) \quad \forall x \in \mathcal{M}.$$

Euler characteristic $\rightsquigarrow$ Betti numbers : refined invariants

* DT in the presence of strictly semistables : Joyce-Song, ...
Kontsevich-Soibelman.

Cohomological integrality conjecture

$\mathcal{E}$ 3CY Abelian category w/ vanishing Euler form.

$\mathfrak{M}_{\mathcal{E}}$ stack of objects in $\mathcal{E}$

$$\mathfrak{M}_{\mathcal{E}} = \bigsqcup_{m \in \pi_0(\mathfrak{M}_{\mathcal{E}})} \mathfrak{M}_{\mathcal{E},m}$$

$\exists\ \mathrm{BPS}_{\mathcal{E},m}$ MHS $\forall m \in \pi_0(\mathfrak{M}_{\mathcal{E}})$

cohomologically graded, bounded $\in \mathcal{D}^b(\mathrm{MHS})$

$\exists$ iso

$$\bigoplus_{m \in \pi_0(\mathfrak{M}_{\mathcal{E}})} H^*(\mathfrak{M}_{\mathcal{E},m}, \mathcal{DT}_m) \simeq \mathrm{Sym}\Big(\bigoplus_{0 \neq m \in \pi_0(\mathfrak{M}_{\mathcal{E}})} \mathrm{BPS}_{\mathcal{E},m} \otimes H^*(B\mathbb{C}^*)\Big)$$

* $\mathcal{E} = \mathrm{Rep}(Q, W)$ $\longrightarrow$ $W = 0$: Meinhardt-Reineke 2014
  $\longrightarrow$ W arbitrary : Davison-Meinhardt 2016.

* $\mathcal{C}$ is 3CY completion of 2CY category [in the sense of Keller]
  e.g. $\mathrm{Coh}_c(S \times \mathbb{A}^1)$ S K3, Abelian, (completion of $\mathrm{Coh}_c(S)$) T^*C

  $\mathrm{Rep}(\tilde{Q}, W)$ tripled quiver with its canonical potential (completion of $\mathcal{G}_2(\mathbb{C}Q)$)

  [Davison - H- Schlegel Mejia]

* $\mathcal{E}$ 3CY : 1st step is to construct a CoHA product on $H(\mathfrak{M}_{\mathcal{E}}, \mathcal{DT})$
  $\rightsquigarrow$ Kinjo - Park - Safronov June 2024.
  extending Kontsevich - Soibelman 2008 quivers w/ potential.

<u>Conclusion</u> : coh. integrality for 1-dim categories, 2 and 3CY categories, is almost fully settled.

<u>Question</u> : Higher dimensional categories, i.e., e.g. 4CY ?
$\rightarrow$ fully open, results regarding virtual structure sheaf (Kool)

<u>Today</u>: Go back at the <u>root</u> of all these results
opposite direction
$\hookrightarrow$ elementary things representation theory.

Object of study

quotient stack

rational coefficients.

$$H^*(V/G) = H^*(V/G, \mathbb{Q})$$

$$\cong H^*(pt/G)$$

$$\cong H^*_G(pt) \cong H^*(BG)$$

G reductive group

V representation.

($=$ vector space V $+$ $G \to GL(V)$).

$\rightsquigarrow$ this is a polynomial ring

$T \subset G$ max torus

$\mathfrak{t} = \mathrm{Lie}(T)$

$W = N_G(T)/T$ Weyl group

$$H^*(V/G) \cong \mathrm{Sym}(\mathfrak{t}^*)^W$$

$$\cong \mathbb{Q}[p_1, \dots, p_{\mathrm{rank}\, G}]$$

$\mathrm{rk}\, G = \dim T$.

The degrees $d_i := \deg p_i$ are well-known and important in rep th

$(\mathbb{C}^*)^n \subset GL_n$ diag matrices

$\mathfrak{S}_n$ symmetric group

$$H^*(pt/GL_n) \cong \mathbb{Q}[x_1, \dots, x_n]^{\mathfrak{S}_n}$$

symm pols in n variables.

$$H^*(pt/GL_2) \cong \mathbb{Q}[x_1 + x_2, x_1 x_2].$$

$d_i = i$.

1 Coh. integrality for symmetric quivers.

At the heart of coh. integrality results is a result of Efimov (2011) I recall now.

$Q = (Q_0, Q_1)$ quiver. $\bullet \to \bullet \circlearrowleft$

Q_0: vertices, Q_1: arrows

$d \in \mathbb{N}^{Q_0}$ dimension vector

$$X_{Q,d} = \bigoplus_{a \in Q_1} \mathrm{Hom}\left(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}}\right)$$

$\circlearrowleft$ changes of basis at vertices

$$GL_d := \prod_{i \in Q_0} GL_{d_i}.$$

$$\mathcal{H}_Q = \bigoplus_{d \in \mathbb{N}^{Q_0}} H^*(X_{Q,d}/GL_d)$$ Kontsevich-Soibelman.

Induction product [defined later in more general context]

$*$ algebra structure on $\mathcal{H}_Q$.

Induction product when $Q =$  g loops.

$$f \in \mathcal{H}_{Q,d} = \mathbb{C}[x_1, \dots, x_d]^{\mathfrak{S}_d}$$
$$g \in \mathcal{H}_{Q,e}$$

$$f * g = \sum_{\sigma \in Sh(d,e)} \sigma \cdot \left\{ f(x_1, \dots, x_d)\, g(x_{d+1}, \dots, x_{d+e}) \cdot \prod_{\substack{1 \leq i \leq d \\ d+1 \leq j \leq d+e}} (x_i - x_j)^{g-1} \right\}$$

~~shuffle algebra product~~ (Feigin - Odesskii)

When Q is symmetric (Q and Q^{op} are the same quivers)

twist $*'$ of $*$ $\qquad *'_{d,e} = (-1)^{\varepsilon(d,e)} *$ +
$\mathbb{Z}$-grading on $\mathcal{H}_Q$ (shifted coh degree) s.t.
$\mathcal{H}_Q$ is supercommutative

Theorem (Efimov 2011) Q **symmetric quiver**

$\exists$ $P \subset \mathcal{H}_Q$, $\mathbb{Z}^{Q_0} \times \mathbb{Z}$- graded and fin. dim pieces, s.t.

$$(\mathcal{H}_Q, *') \cong \mathrm{Sym}\Big(P \otimes \underbrace{\mathbb{Q}[x]}_{\text{coh. degree } 2\ (\mathbb{Z})}\Big)$$

as supercommutative algebras

Theorem (Meinhardt–Reineke, 2014)

Let $d \in \mathbb{N}^{Q_0}$.

$$P_d = \begin{cases} IH^*(X_{Q,d} /\!/ GL_d) & \text{if } Q \text{ has simple representations of dim } d \\ 0 & \text{otherwise.} \end{cases}$$

$\rightsquigarrow$ categorification of Efimov's identity.

$\rightsquigarrow$ very powerful way to compute $IH^*(X_{Q,d} /\!/ GL_d)$.

More on today: • Explain that Efimov's result is a very particular case of something much more general.

• First part of my research project whose aim is the study of enumerative invariants of smooth stacks, 0- and (−1)-shifted symplectic stacks.

2CY cat

3CY category

② Symmetric representations of reductive groups: example

examples: decompositions of polynomial rings.

2, 0, −2
$3/2 = 1{,}5$
1, 3
$\left\lfloor \frac{d-1}{2} \right\rfloor$

ⓐ $SL_2(\mathbb{C}) \curvearrowright \mathbb{C}^d$ irreducible representation.

$$\mathbb{Q}[x^2]_{\deg < \left\lfloor \frac{d-1}{2} \right\rfloor} \oplus \mathbb{Q}[x]^{\varepsilon} \to \mathbb{Q}[x^2]$$

$$(f, g) \mapsto f + \underset{\substack{\cap \\ \mathbb{Q} \text{ constant}}}{c_d}\, x^{\left\lfloor \frac{d-1}{2} \right\rfloor} g$$

isomorphism.

$\mathbb{Q}[x]^{\varepsilon} = \mathbb{Q}[x^2]$ if $\left\lfloor \frac{d-1}{2} \right\rfloor$ even
$= x\,\mathbb{Q}[x^2]$ if odd.

ⓑ $GL_2 \curvearrowright \mathbb{C}^2 \oplus (\mathbb{C}^2)^{\vee} = T^*\mathbb{C}^2$

We can write an isomorphism

$$\mathbb{Q}[x_1] \oplus \mathbb{Q}[x_1, x_2]^{\mathrm{sgn}} \longrightarrow \mathbb{Q}[x_1, x_2]^{\mathfrak{S}_2} \cong \mathbb{Q}[\underset{\substack{\| \\ y}}{x_1 + x_2}, \underset{\substack{\| \\ z}}{x_1 x_2}]$$

$$f(x_1), g(x_1, x_2) \longmapsto \frac{x_1 f(x_1) - x_2 f(x_2)}{x_1 - x_2} + \underbrace{2 x_1 x_2 \frac{g(x_1, x_2)}{x_1 - x_2}}_{z\,\mathbb{Q}[y, z]}$$

$$f = x^k \rightsquigarrow \frac{x_1^{k+1} - x_2^{k+1}}{x_1 - x_2} = x_1^k + x_1^{k-1} x_2 + \dots + x_2^k$$

$$\sim \underset{\substack{\| \\ y^k}}{(x_1 + x_2)^k} \text{ modulo } z\,\mathbb{Q}[y, z].$$

(2) Symmetric representations of reductive groups

Q symm quiver, d dim vect $\rightsquigarrow$ $X_{Q,d} \circlearrowleft GL_d$ symmetric representation.

Objective: get rid of the quiver context.

→ much more general

→ adapted to the study of general smooth stacks

G reductive group

V representation

$T \subset G$ maximal torus

$$V = \bigoplus_{\alpha : \mathbb{G}_m \to T} V_\alpha \qquad V_\alpha = \left\{ v \in V \mid t \cdot v = \alpha(t) v \ \forall t \in T \right\}$$

α s.t. $V_\alpha \neq 0$ $\rightsquigarrow$ weight of V.

$\mathcal{W}(V)$ = weights of V counted with multiplicities

Def: V is called symmetric if $\mathcal{W}(V) = \mathcal{W}(V^*)$.

ex:
- T^*W for any representation W of G
- $\mathfrak{g}$ with adjoint action.
- Symm. reps are stable under various operations: $\oplus, \otimes, \dots$

Parabolic induction

$X^*(T) = \{T \longrightarrow \mathbb{G}_m\}$ characters

$X_*(T) = \{\mathbb{G}_m \to T\}$ cocharacters

$X_*(T) \times X^*(T) \longrightarrow \mathbb{Z}$ pairing

$(\lambda \, , \, \alpha) \longmapsto \langle \lambda, \alpha \rangle$

$\alpha \circ \lambda \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}.$

let $\lambda \in X_*(T)$

$$V^\lambda = \{ v \in V \mid \lambda(t) \cdot v = v \ \forall t \in \mathbb{G}_m \}$$

$$= \bigoplus_{\substack{\alpha \in X^*(T) \\ \langle \lambda, \alpha \rangle = 0}} V_\alpha$$

$$V^{\lambda \geq 0} = \{ v \in V \mid \lim_{t \to 0} \lambda(t) \cdot v \text{ exists} \}$$

$$= \bigoplus_{\substack{\alpha \in X^*(T) \\ \langle \lambda, \alpha \rangle \geq 0}} V_\alpha$$

$$G^{\lambda} = \{ g \quad G \mid \lambda(t)\, g\, \lambda(t)^{-1} = g \quad \forall t \in \mathbb{G}_m \}$$

Levi subgroup of G

$$G^{\lambda \geq 0} = \{ g \in G \mid \lim_{t \to 0} \lambda(t)\, g\, \lambda(t)^{-1} \text{ exists} \}$$

$\subset G$ parabolic subgroup

$G^{\lambda \geq 0} \to G^{\lambda}$ limit morphism.

V^{λ} is a G^{λ}-representation

$V^{\lambda \geq 0}$ is a $G^{\lambda \geq 0}$-representation

Induction diagram

$$\begin{tikzcd}
 & V^{\lambda \geq 0}/P^{\lambda \geq 0} \arrow[dl, "q_\lambda"'] \arrow[dr, "p_\lambda"] & \\
V^{\lambda}/G^{\lambda} & & V/G
\end{tikzcd}$$

diagram of stacks

Proposition ① q_λ is smooth (not representable! but not important)

② p_λ is representable and projective

Proof

① q_λ comes from the equivariant map

$$\left(V^{\lambda\geq 0}, P^{\lambda\geq 0}\right) \to \left(V^\lambda, G^\lambda\right)$$

and $V^{\lambda\geq 0} \to V^\lambda$ is a vector bundle.

② $$V^{\lambda\geq 0}/P^{\lambda\geq 0} \simeq G \times^{P^{\lambda\geq 0}} V^{\lambda\geq 0} \big/ G$$

where $$G \times^{P^{\lambda\geq 0}} V^{\lambda\geq 0} := G \times V^{\lambda\geq 0} \big/ P^{\lambda\geq 0}$$ diagonal action.

and
$$\begin{tikzcd}
G \times^{P^{\lambda\geq 0}} V^{\lambda\geq 0} \arrow[r] \arrow[dr, hook, "\text{closed imm.}"] & V \\
& G/P^{\lambda\geq 0} \times V \arrow[u]
\end{tikzcd}$$
is projective

$G/P^{\lambda\geq 0}$ projective variety.

$$(g, v) \mapsto \left(gP^{\lambda\geq 0}, g\cdot v\right)$$

$$q_\lambda^* : H^*(V^\lambda / G^\lambda) \longrightarrow H^*(V^{\lambda \geq 0} / P^{\lambda \geq 0})$$

$$(p_\lambda)_* : H^*(V^{\lambda \geq 0} / P^{\lambda \geq 0}) \to H^*(V/G)$$

$$\mathrm{Ind}_\lambda := (p_\lambda)_* \circ (q_\lambda)^* : H^*(V^\lambda / G^\lambda) \longrightarrow H^*(V/G).$$

induction map.

$$k_\lambda := \frac{\prod_{\alpha \in W(V),\, \langle \lambda, \alpha \rangle > 0} \alpha}{\prod_{\alpha \in W(\mathfrak{g}),\, \langle \lambda, \alpha \rangle > 0} \alpha}$$

Proposition (explicit formula) $f \in H^*(V^\lambda / G^\lambda)$.

$$\mathrm{Ind}_\lambda(f) = \frac{1}{\# W^\lambda} \sum_{w \in W} w \cdot (f k_\lambda) \in H^*(V/G)$$

polynomial.

W^λ = Weyl group of G^λ

Proof Localisation of equivariant cohomology + computation of Euler classes.

Cohomological integrality

<u>equivalence relation</u> :

$$\lambda, \mu \in X_*(T)$$

$$\lambda \sim \mu \iff \begin{cases} V^\lambda = V^\mu \\ G^\lambda = G^\mu \end{cases}$$

$$\mathcal{P}_V := X_*(T)/\sim$$

infinite

finite

$\lambda \in X_*(T) \rightsquigarrow \bar{\lambda} \in \mathcal{P}_V$.

Weyl group of G : $W = N_G(T)/T$ finite group

Coxeter group

$W \curvearrowright X_*(T)$: $\mathbb{G}_m \xrightarrow{\lambda} T$

$w \in W$, $\dot{w} \in N_G(T)$ lift

$w \cdot \lambda(t) = \dot{w}\, \lambda(t)\, \dot{w}^{-1}$

$W \curvearrowright \mathcal{P}_V$: the action descends

For $\lambda \in X_*(T)$, $W_\lambda := \{ w \in W, \ \overline{w \cdot \lambda} = \bar{\lambda} \}$.

$G_\lambda := \ker\left(G^\lambda \to GL(V^\lambda)\right) \cap \underbrace{Z(G^\lambda)}_{\text{centre}}$

Theorem (H. 2024)

$\exists$ characters $\varepsilon_{V,\lambda} : W_\lambda \to \{\pm 1\}$ such that

$\exists$ finite dimensional $\mathcal{P}_\lambda \subset H^*(V^\lambda / G^\lambda)$

$\exists$ W_λ-action on $\mathcal{P}_\lambda$

only keep the $\varepsilon_{V,\lambda}$-isotypic component inside $\mathcal{P}_\lambda \otimes H^*(pt/G_\lambda)$

s.t.

$$\bigoplus_{\tilde{\lambda} \in \mathcal{P}_V / W} \left(\mathcal{P}_\lambda \otimes H^*(pt/G_\lambda)\right)^{\varepsilon_{V,\lambda}} \xrightarrow{\bigoplus \mathrm{Ind}_\lambda} H^*(V/G)$$

is an isomorphism.

Character $\varepsilon_{V,\lambda}$: what replaces twist of CoHA mult in Efimov's paper. $w \in W_\lambda$. $w(k_\lambda) = (-1)^{\varepsilon_{V,\lambda}(w)} k_\lambda$ defines it.

Proof (Steps)

1. Find P_λ (easy)
2. Surjectivity holds by construction
3. Show P_λ is finite dimensional (medium difficulty)
4. Show injectivity (hard difficulty)

① Find P_λ $\qquad \lambda =$ triv .

$$H^*(V/G) \cong \underbrace{H^*\big(V/(G/G_0)\big)}_{\mathcal{H}^{prim}} \otimes H^*(pt/G_0)$$

$$\mathcal{A}^{prim} : H^*\big(V/(T/G_0)\big) \qquad \text{so that } \mathcal{H}^{prim} = \left(\mathcal{A}^{prim}\right)^W .$$

$J_0 :=$ smallest W-invariant $\mathcal{A}^{prim}$ submod of $\mathcal{A}^{prim}\Big[\prod_{\alpha \in \mathcal{W}(\mathfrak{g})\setminus\{0\}} \alpha^{-1}\Big]$

containing $k_\lambda = \dfrac{\prod_{\alpha \in \mathcal{W}(V),\, \langle \lambda, \alpha \rangle > 0} \alpha}{\prod_{\alpha \in \mathcal{W}(\mathfrak{g}),\, \langle \lambda, \alpha \rangle > 0} \alpha}$ $\quad \forall \lambda \in P_V$, $\lambda \neq \bar{0}$.

$J_0^W \subset \mathcal{H}^{prim}$

P_0 = direct sum complement of J_0^W. ∎