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deformed Calabi-Yau completion and its application to DT theory

0. Introduction

non-commutative algebraic geometry (NCAG)

… study of dg-categories.

e.g

- X : scheme. $Qcoh(X)$ … dg-cat of quasi-coherent complexes on X.
- A : dg-algebra Mod_A : dg-cat of A-modules. //

AG $\subset$ NCAG $\rightsquigarrow$ Representation theory.

CY mfd $\subset$ CY category play an important role.

deformed CY completion

$\mathcal{C}$: dg-category, $n \in \mathbb{Z}$, $c \in HC^{-}_{n-1}(\mathcal{C})$.

keller $\rightsquigarrow$ $\Pi_{n+1}(\mathcal{C}, c)$: deformed CY completion of $\mathcal{C}$.

Y : sm var of dim $= d$

eg $(c=0)$

- (RT) preprojective algebra $(n=2)$
- (AG) Total space of the canonical bundle. $(n = \dim + 1)$

$\Pi_{d+1}(\mathrm{Qcoh}(Y))$

$\simeq \mathrm{Qcoh}(\mathrm{Tot}_Y(\omega_Y))$

$(c \neq 0)$

- (RT) Ginzburg dg-algebra $(n=3)$
- (AG) Affine bundle modeled on the canonical bundle.

$W \in HH_0(\mathbb{C}Q)$: potential

$\delta w \in HC_1^-(\mathbb{C}Q)$

$HH_{d-1}(Y) \supset H^1(Y, \omega_Y)$

We will see def CY completion has a striking application in DT theory of Higgs bundles //

dim reduction.

§1. deformed CY completion.

$k = \bar{k}$, $\mathrm{ch}(k) = 0$.

$\mathcal{C}$: finite type presentable DG category.

(i.e $\mathcal{C} \simeq \mathrm{Mod}_R$, R : homotopically finitely presented dg-algebra.)

$\mathrm{Id}_{\mathcal{C}}^! : \mathcal{C} \longrightarrow \mathcal{C}$: inverse dualizing functor.

$$\left(\begin{array}{ll} \mathrm{Hom}(E,F) \simeq \mathrm{Hom}(\mathrm{Id}_{\mathcal{C}}^!(F), E)^\vee & \\ E : \text{compact} \quad F : \text{right proper} & \end{array} \right)$$

$$\Pi_n(\mathcal{C}) := \mathrm{Mod}_{\mathrm{Free}(\mathrm{Id}_{\mathcal{C}}^![n-1])}(\mathcal{C})$$

free monad generated by $\mathrm{Id}_{\mathcal{C}}^![n-1]$

monad

= Algebra object in $\mathrm{End}(\mathcal{C})$

$A \in \mathrm{Mnd}(\mathcal{C})$

$\left(\begin{array}{l} obj(\Pi_n(\mathcal{C})) = \left\{ (E,\phi) \;\middle|\; \begin{array}{l} E \in \mathcal{C} \\ \phi : Id_{\mathcal{C}}^{!}(E)[n-1] \to E \end{array} \right\} \\ \text{1-morph : morph of obj in } \mathcal{C} + \text{compatibility data.} \end{array} \right)$

A-mod in $\mathcal{C}$ is $E \in \mathcal{C}$, $A \otimes E \to E$, ... //

e.g. • [keller] $\mathcal{C} := Mod_Q$ Q : non-Dynkin quiver.

$\Pi_2(Mod_Q) \simeq Mod_{\Pi_2(Q)}$ $\Pi_2(Q)$: preprojective algebra.

• [Ikeda-Qiu]

Y : sm variety of $\dim = d$

$\Pi_{d+1}(Qcoh(Y)) = Qcoh(Tot_Y(\omega_Y))$. //

deformed case

$c \in HH_{d-1}(\mathcal{C}) = Hom(Id_{\mathcal{C}}^{!}[d-1], Id_{\mathcal{C}})$

$c : Id_{\mathcal{C}}^{!}[d-1] \to Id_{\mathcal{C}}$

universality $\rightsquigarrow$ $Free(Id_{\mathcal{C}}^{!}[d-1]) \to Id_{\mathcal{C}}$: morphism of monad.

$Mod_{-}(\mathcal{C})$ $\rightsquigarrow$ $\Pi_d(\mathcal{C}) \xrightarrow{i_c^*} \mathcal{C}$

$$\begin{array}{ccc} \Pi_d(\mathcal{C}) & \xrightarrow{i_c^*} & \mathcal{C} \\ {\scriptstyle i_0^*}\downarrow & \ulcorner & \downarrow \\ \mathcal{C} & \longrightarrow & \Pi_{d+1}(\mathcal{C},c) \end{array}$$

deformed CY completion.

$\mathcal{C} = Qcoh(Y)$
$\dim Y = d$.
$H^0(Y, \omega_Y) \subset HH_d(Y)$.
$c \in H^0(Y, \omega_Y)$,
$i_c^* : Qcoh(Tot_Y(\omega_Y)) \to Qcoh(Y)$
pullback of the section.

$\boxed{\text{Rem}}$ We need a lift $\tilde{c} \in HC^-(\mathcal{C})$ to construct a CY-str on $\Pi_{d+1}(\mathcal{C},c)$.

Yeung gave an example

e.g. • (RT)

Q : quiver $W \in HH_0(\mathbb{C}Q)$: potential.

$\Pi_3(\mathrm{Rep}_Q, Sw) \simeq \mathrm{Rep}_{\Gamma(Q,w)}$ $\Gamma(Q,w)$: Ginzburg dg-algebra.

- (AG)

  Y : sm var of $\dim = d$. $c \in H^1(Y, \omega_Y) \subset HH_{d-1}(Y)$.

  X : ω_Y-torsor corresponding to c.

<u>Thm</u> (K-Masuda)

equivalence of CY category.

$$\Pi_{d+1}(\mathrm{Qcoh}(Y), c) \simeq \mathrm{Qcoh}(X).$$

<u>idea</u> (far from accurate)

"$\mathrm{Spec}(\mathrm{Free}(\omega_Y^\vee[-1]))$"

Z : nc scheme.

- $\Pi_d(\mathrm{Qcoh}(Y)) \simeq \mathrm{Qcoh}(\cancel{[Y/\omega_Y]})$

- $$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "c"] \\ Y \arrow[r, "0"] & \cancel{[Y/\omega_Y]} \; Z \end{tikzcd} \rightsquigarrow \begin{tikzcd} \Pi_d(\mathrm{Qcoh}(Y)) \arrow[r] \arrow[d] & \mathrm{Qcoh}(Y) \arrow[d] \\ \mathrm{Qcoh}(Y) \arrow[r] & \mathrm{Qcoh}(X) \end{tikzcd}$$

  pullback of nc scheme

§2. moduli of objects. (Bozec-Calaque-Scherotzke).

$\mathcal{C}$: finite type presentable DG-category.

$\mathfrak{M}_{\mathcal{C}}$: moduli stack of objects in $\mathcal{C}$.

$x \in \mathfrak{M}_{\mathcal{C}}(k) \xleftrightarrow{1:1} x \in \mathcal{C}^{\text{right-proper}}$.

cotangent complex

$$\mathbb{L}_{\mathfrak{M}_{\mathcal{C}}}|_x = (\mathrm{Hom}(x,x)[1])^\vee.$$

<u>Thm</u> (Bozec-Calaque-Scherotzke)

$$\mathfrak{M}_{\Pi_d(\mathcal{C})} \simeq T^*[2-d]\mathfrak{M}_{\mathcal{C}}$$
$$(= \mathrm{Tot}(\mathbb{L}_{\mathfrak{M}_{\mathcal{C}}}[2-d])).$$

idea

$p : \mathfrak{M}_{\pi_d(\mathcal{C})} \longrightarrow \mathfrak{M}_{\mathcal{C}}.$

$p^{-1}(x) = \mathrm{Hom}(\mathrm{Id}_{\mathcal{C}}^{!}(x)[d-1], x)$

$\overset{\text{Serre}}{\simeq} \mathrm{Hom}(x, x[d-1])^{\vee}$

$\simeq (\mathbb{L}_{\mathfrak{M}_{\mathcal{C}}}[2-d])|_x.$

$c \in HH_0(\mathcal{C})$

$x \in \mathcal{C}^{\text{right-proper}} \simeq \mathrm{Funct}(\mathcal{C}^c, \mathrm{Vect}_k^{\text{fin-dim}})$

$f_c(x) := HH_0(x)(c) \in HH_0(\mathrm{Vect}_k) \simeq k.$

$f_c \in \Gamma(\mathfrak{M}_{\mathcal{C}}, \mathcal{O}_{\mathfrak{M}_{\mathcal{C}}}).$

Thm (BCS)

$\mathfrak{M}_{\pi_3(\mathcal{C}, Sc)} \simeq \mathrm{Crit}(f_c).$ //

$S : HH_0(\mathcal{C}) \to HC_1^{-}(\mathcal{C})$

idea

$$\begin{tikzcd} \pi_2(\mathcal{C}) \arrow[r, "i_c^*"] \arrow[d, "i_0^*"'] & \mathcal{C} \arrow[d] \\ \mathcal{C} \arrow[r] & \pi_3(\mathcal{C}, c) \end{tikzcd} \longleftrightarrow \begin{tikzcd} \mathfrak{M}_{\pi_3(\mathcal{C}, Sc)} \arrow[r] \arrow[d] & \mathfrak{M}_{\mathcal{C}} \arrow[d, "0"] \\ \mathfrak{M}_{\mathcal{C}} \arrow[r, "df_c"'] & T^*\mathfrak{M}_{\mathcal{C}} \end{tikzcd}$$

$3 = \underbrace{2}_{\mathcal{C}} + 1$ 0-shifted cotangent.

§3. Application to DT theory and Higgs bundles.

X : CY 3-fold H : ample div on X

For $\beta \in H^2(X)$ and $d \in \mathbb{Z}$,

$\mathfrak{M}^{ss}_{X,\beta,m}$: moduli stack of semi-stable sheaves w/ 1-dim support w/ $ch_1(E) = \beta$, $\chi(E) = m$.

fix an orientation data

$\varphi_{\mathfrak{M}^{ss}_{X,\beta,m}} \in \mathrm{Perv}(\mathfrak{M}^{ss}_{X,\beta,m})$: Joyce's perverse sheaf.

$H^*(\mathfrak{M}^{ss}_{X,\beta,m}, \varphi_{\mathfrak{M}^{ss}_{X,\beta,m}})$: categorification of DT invariant.

$$DT_\beta = \sum_m \frac{1}{m^2} \widehat{DT}_{\frac{\beta}{m}}$$

Thm (K-Koseki)

$r \in \mathbb{Z}_{>0}$: rank $m \in \mathbb{Z}$

C : sm proj curve

N : rank two vector bundle $\det(N) \simeq \omega_C$.

$0 \to L_1 \to N \to L_2 \to 0$: short exact seq, $\deg L_2 >> 2g(C)-2$.

local curve (CY 3-fold)

$X := \mathrm{Tot}_C(N)$ $Y := \mathrm{Tot}_C(L_2)$

$\exists f : \mathfrak{M}^{ss}_{Y,r,m} \to \mathbb{A}^1$ s.t. $\Rightarrow \mathfrak{M}^{ss}_{Y,r,m}$: smooth.

- $\mathfrak{M}^{ss}_{X,r\cdot[C],m} \simeq \mathrm{Crit}(f)$
- $\varphi_{\mathfrak{M}^{ss}_{X,r,m}} \simeq \varphi_f(\mathbb{Q}_{\mathfrak{M}^{ss}_{Y,r,m}}[-])$. (w.r.t. certain orien data)

idea

- $X \to Y$: ω_Y-torsor.
- $\alpha \in H^1(Y, \omega_Y)$: corr to this torsor.

  One can show $\alpha = \delta_C$ ($C \in HH_0(Y)$).
- Use [BCS], we obtain the claim

Application.

X: CY3-fold $\widehat{DT}_{\beta,m}(X)$: DT inv counting semistable coh sheaves on X w/ $ch_1(E)=\beta$, $\chi(E)=m$.

$JH_X : \mathfrak{M}^{ss}_{X,r,m} \longrightarrow M^{ss}_{X,r,m}$: morphism to the coarse moduli.

$$\varphi_{M^{ss}_{X,r,m}} := {}^p\mathcal{H}^1((JH_X)_*\varphi_{\mathfrak{M}^{ss}_{X,r,m}}).$$

$$\widehat{DT}_{\beta,m}(X) := \chi(M^{ss}_{X,r,m}, \varphi_{M^{ss}_{X,r,m}}).$$

$B\mathbb{C}^* \xrightarrow{p} pt$

$\varphi_{B\mathbb{C}^*} = \mathbb{Q}_{B\mathbb{C}^*}[-1]$

${}^p\mathcal{H}^1(p_*\mathbb{Q}_{B\mathbb{C}^*}[-1]) = \mathbb{Q}$.

Conj (Joyce-Song, Toda, χ-independence)

$\widehat{DT}_{\beta,m} = \widehat{DT}_{\beta,m'}$ for β, m, m'.

Rmk

- χ-indep $\Longleftrightarrow$ strong rationality: $\sum PT_{\beta,n} t^n = \prod (\quad)$ ← Pandharipande-Thomas inv; function of GV.
- known for quintic — PT/GW corresp + GV finiteness conj. Doan-Ionel-Walpuski: compact case

... finiteness conj for GW inv ... invariant)

$\Rightarrow$ χ - independence.

Thm (K-Koseki)

χ-indep conj is true for $\mathrm{Tot}_C(H)$.

idea

χ-indep conj + IC cohomology

- χ-indep conj is known for $\mathrm{Tot}_C(L_2)$. $(\deg L_2 >> 2g-2)$.

  (Maulik-Shen, Ngo's support thm.)

- Applying the vanishing cycle functor.