TD 6 -

$$\left\{\begin{array}{c}\text{simply-connected}\\ \text{Poisson-Lie groups}\end{array}\right\} \xrightarrow[\sim]{F} \{\text{Lie bialgebra}\}$$
$$G \longmapsto \mathrm{Lie}(G)$$

$$\{\text{s.c. Lie group}\} \xrightarrow{\sim} \{\text{Lie algebras}\}$$
$$G \longmapsto \mathrm{Lie}\, G$$

exc 6.1 : $\underset{\text{Borel}}{\mathfrak{b}} \subset \mathfrak{g}^{\text{fin dim}}$ $\qquad N_{\mathfrak{g}}(\mathfrak{b}) = \{x \in \mathfrak{g} \mid [x,\mathfrak{b}] \subset \mathfrak{b}\} \supset \mathfrak{b}$

want an equality.

$\mathfrak{b}$ is a maximal solvable Lie subalgebra of $\mathfrak{g}$

($\mathfrak{r}$ = radical ——— ideal ——)

It suffices to show that $N_{\mathfrak{g}}(\mathfrak{b})$ is a solvable Lie subalgebra of $\mathfrak{g}$.

$$0 \to \underbrace{\mathfrak{b}}_{\text{solvable}} \to N_{\mathfrak{g}}(\mathfrak{b}) \to \underbrace{N_{\mathfrak{g}}(\mathfrak{b})/\mathfrak{b}}_{\text{solvable!}} \to 0$$

$\mathfrak{b} \subset N_{\mathfrak{g}}(\mathfrak{b})$ solvable ideal maximal. $\mathfrak{b}$ is the radical of $N_{\mathfrak{g}}(\mathfrak{b})$. so $N_{\mathfrak{g}}(\mathfrak{b})/\mathfrak{r}$

2- $\mathfrak{b} \subset \mathfrak{g}$, $\qquad \mathfrak{r}$ is the unique max. solvable ideal of $\mathfrak{g}$.

$\exists\, \mathfrak{b}' \supset \mathfrak{r}$ Borel subalgebra containing $\mathfrak{r}$

- $g \in \mathrm{Aut}(\mathfrak{g})$, $\quad g(\mathfrak{r})$ is again a maximal solvable ideal of $\mathfrak{g}$.

  so $\mathfrak{r} = g(\mathfrak{r})$.

- $\exists\, g \in \mathrm{Aut}(\mathfrak{g})$ s.t. $g(\mathfrak{b}') = \mathfrak{b}$. So $\qquad g(\mathfrak{b}') = \mathfrak{b} \supset g(\mathfrak{r}) = \mathfrak{r}$.

1- $y \in N_{\mathfrak{g}}(\mathfrak{b}) \setminus \mathfrak{b}$, $\qquad \mathfrak{b}' = \mathfrak{b} \oplus \mathbb{C}y$

$[\mathfrak{b}', \mathfrak{b}'] \subset \mathfrak{b}$ $\quad$ so $\mathfrak{b}'$ is a solvable Lie algebra of $\mathfrak{g}$

$\mathfrak{b} \subsetneq \mathfrak{b}'$: impossible by maximality of $\mathfrak{b}$.

comm assoc alg

ex 6.2: $(A, \{-,-\})$ $\quad \{-,-\} : A^{\otimes 2} \longrightarrow A$

Lie bracket.

Leibniz rule:

$(M, \{-,-\})$ Poisson manifold.

1- M Poisson $\quad f, g \in \mathcal{C}^\infty(M)$

$\forall x \in M, \quad \{f,g\}(x) = \left(df(x) \otimes dg(x)\right)\Pi(x)$

M algebraic variety, $U \subset M$, $f, g \in \Gamma(U, \mathcal{O}_M)$.

$$\Pi = \sum_{i,j} \Pi_{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} \quad \text{in local coordinates.}$$

- derivations of $\mathcal{C}^\infty(M)$ $\longleftrightarrow$ vector fields on M

biderivation: lin map $b : \mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ $\longleftrightarrow$ bivector fields on M.

satisfying Leibniz rule

$$b(ab, c) = a\,b(b,c) + b\,b(a,c)$$
$$b(a, bc) = b(a,b)c + b(a,c)b$$

- derivations are local: $D : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ derivation

$Df(x)$ only depends on $f|_U$, $U \ni x$ arbitrary small neighbourhood.

$$\rho \in \mathcal{C}^\infty(M) \qquad \rho(x) = 1, \quad \rho|_{M \setminus U} \equiv 0$$

If $f, g \in \mathcal{C}^\infty(M)$ st $f|_U = g|_U$,

$$\rho(f-g) \equiv 0$$

$$0 \equiv D(\rho(f-g)) = D\rho\,(f-g) + \rho\, D(f-g)$$

So

$$0 = (D\rho)\cancel{(x)}(f(x) - g(x)) + \underset{\overset{\shortparallel}{1}}{\rho(x)}\left(Df(x) - Dg(x)\right)$$

$\rightarrow$ Assume $M = U \subset \mathbb{R}^n$ convex.

$\forall x, y \in U$, consider $t \overset{\varphi}{\longmapsto} f(y + t(x-y))$
$[0,1] \longrightarrow \mathbb{R}$

$$\varphi(1) = \varphi(0) + \int_0^1 \varphi'(t)\,dt$$

$$f(x) = f(y) + \int_0^1 \underbrace{df\big(y+t(x-y)\big)(x-y)}_{\sum_{i=1}^n \frac{\partial f}{\partial x_i}(y+t(x-y))(x_i-y_i)}\,dt.$$

$\boxed{y \text{ is fixed}}$

$$= f(y) + \sum_{i=1}^n (x_i - y_i) \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}\big(y+t(x-y)\big)\,dt}_{g_i(x)}$$

$x_i : U \longrightarrow \mathbb{R}$
$(x_1, \ldots, x_n) \longmapsto x_i$

$$g_i(y) = \frac{\partial f}{\partial x_i}(y).$$

$$Df(x) = \sum_{i=1}^n D(x_i - \cancel{y_i})\, g_i(x) + \sum_{i=1}^n (x_i - y_i)(D g_i)(x)$$

$x = y$ $\quad Df(y) = \sum_{i=1}^n D(x_i) \frac{\partial f}{\partial x_i}(y).$

$$= X_D \cdot f(y) \quad \text{where} \quad X_D = \sum_{i=1}^n D(x_i) \frac{\partial}{\partial x_i}$$
$$= df(y)\big(X_D(y)\big)$$

for $\{-,-\}$ (biderivation) : do the same thing.

$$f(x) = f(y) + \sum_i (x_i - y_i)\, g_i(x)$$
$$g(x) = g(y) + \sum_i (x_i - y_i)\, h_i(x)$$

$\{f, g\}$ . . .

$$\Pi = \sum_{i,j=1}^n \{x_i, x_j\} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} = \sum \Pi_{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}$$

$$\Pi \in \Gamma(TM^{\otimes 2})$$

$\{-,-\}$ Poisson bracket $\quad \{g, f\} = -\{f, g\}.$

$$\Rightarrow \Pi \in \Gamma(\Lambda^2 TM). \Rightarrow \Pi_{ij} = -\Pi_{ji}.$$

Jacobi for $\{-,-\}$ $\Rightarrow$ property of π.

$\Updownarrow$

Get : $0 = \sum_j \left(\pi_{ij} \frac{\partial \pi_{kl}}{\partial x_j} + \pi_{kj} \frac{\partial \pi_{li}}{\partial x_j} + \pi_{lj} \frac{\partial \pi_{ik}}{\partial x_j} \right) = \pi_{i,k,l}$

$\forall\, i,k,l.$

trivector field $\left(\in \Gamma(TM^{\otimes 3}) \right)$

$$[\pi,\pi]_S = \sum_{i,j,k} \pi_{i,k,l} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_k} \otimes \frac{\partial}{\partial x_l}.$$

"Schouten bracket".

"Can define $[\pi,\pi']$ for any $\pi, \pi' \in \Gamma(\Lambda^2 TM)$."

$\cap$

$\Gamma(\Lambda^3 TM)$.

3- A symplectic manifold has a canonical Poisson structure.

$\{-,-\}, \pi$

symplectic manifolds are nondegenerate Poisson manifolds.

$\not\Leftarrow$

Symplectic manifold : (M,ω) ω is a closed non degenerate differential 2-form on M, $d\omega = 0$.

$\omega \in \Gamma(\Lambda^2 \Omega^1(M))$

locally, $\omega = \sum_{i<j} \omega_{ij}\, \underbrace{dx_i \wedge dx_j}$ $dx_i \wedge dx_j$, $i<j$ local base of $\Lambda^2 \Omega^1(M)$.

$dx_i \otimes dx_j - dx_j \otimes dx_i$.

non degenerate:

cotangent bundle

$TM \longrightarrow T^*M$

$v \in T_xM \longmapsto \omega_x(v,-) \in T_x^*M$ $\qquad \omega_x : TM \otimes TM \to \mathbb{R}$ antisymmetric

$\hookrightarrow$ this map is an isomorphism.

easiest ex: $\mathbb{R}^{2n}$, ω constant antisymmetric 2-form

sympl. vector space

given by the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = S$

$$\mathbb{R}^{2n} = \underbrace{\mathbb{R}^n}_{x_i} \times \underbrace{\mathbb{R}^n}_{q_i}$$

$$\omega\left((\underline{x},\underline{p}),(\underline{x}',\underline{p}')\right) = (\underline{x},\underline{p})\, S \begin{pmatrix} x'^T \\ p'' \end{pmatrix} = \sum_i x_i p'_i - \sum_i x'_i p_i$$

Next ex: M smooth manifold

T^*M is a symplectic manifold.

locally, $M \simeq \mathbb{R}^n$

$T^*M \simeq \mathbb{R}^{2n}$.

- $g, f \in \mathcal{C}^\infty(M)$ $\qquad \{f,g\} = \omega(X_f, X_g) \in \mathcal{C}^\infty(M)$

$$TM \xrightarrow{\omega} T^*M$$
$$v \longmapsto \omega(v,-)$$

$X_f \qquad df \in \Gamma(T^*M)$

s.t. $df = \omega(X_f, -)$
$= \iota_{X_f}\omega \quad \in \Omega^1 M.$

$$\{f,g\}(x) = \omega_x(X_f(x), X_g(x)) = df(x)(X_g(x)) = X_g \cdot f(x)$$
$$= -\omega_x(X_g(x), X_f(x))$$
$$= -dg(x)(X_f(x)) = -X_f \cdot g(x)$$

- antisymmetry : clear.
- Leibniz rule : $\{fg, h\} = X_h \cdot (fg)$
$= f X_h \cdot g + g X_h \cdot f$
$= f\{g,h\} + g\{f,h\}$. ok.

Jacobi: locally $M \simeq \left(\mathbb{R}^{2n}, \omega = \sum_{i=1}^n dx_i \wedge dp_i\right)$.

(Darboux theorem)

Poisson bracket in local coordinates

$$X = \sum_i \left(X_{x_i}\frac{\partial}{\partial x_i} + X_{p_i}\frac{\partial}{\partial p_i}\right) \qquad Y = \sum_{i=1}^n \left(Y_{x_i}\frac{\partial}{\partial x_i} + Y_{p_i}\frac{\partial}{\partial p_i}\right)$$

$$\omega(Y,X) = \sum \left(Y_{x_i} X_{p_i} - Y_{p_i} X_{x_i}\right).$$

$$df = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial p_i} dp_i \right)$$

(dx_i, dp_i via base of $(\mathbb{R}^{2n})^*$ dual to the base $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial p_i}$ of $\mathbb{R}^{2n}$)

$$df(X) = \sum_i \left(X_{x_i} \frac{\partial f}{\partial x_i} + X_{p_i} \frac{\partial f}{\partial p_i} \right)$$

$df(X) = \omega(Y, X)$ will give $Y = X_f$.

$$X_f = \sum_i \left(\underbrace{\frac{\partial f}{\partial p_i}}_{Y_{x_i}} \frac{\partial}{\partial x_i} - \underbrace{\frac{\partial f}{\partial x_i}}_{Y_{p_i}} \frac{\partial}{\partial p_i} \right)$$

$$\{f, g\} = -dg(X_f) \qquad dg = \sum_i \left(\frac{\partial g}{\partial x_i} dx_i + \frac{\partial g}{\partial p_i} dp_i \right)$$

$$= -\sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} \right).$$

$\{f, \{g, h\}\}$ + circ. permutations $= 0$, : Jacobi.

Always $\quad T^*M \longrightarrow TM \qquad \Pi_x = \sum_i v_i \otimes w_i \in T_xM \otimes T_xM.$

$$l \in T_x^*M \longmapsto \sum_{i=1}^{n} l(v_i) w_i \in T_xM$$

Π nondegenerate $\Leftrightarrow$ isomorphism

4- $\mathfrak{g}$ f.d Lie algebra. $\mathfrak{g}^*$ its dual.

$df(x) : T_x\mathfrak{g}^* \xrightarrow{\text{linear}} \mathbb{C}$, $T_x\mathfrak{g}^* = \mathfrak{g}^*$

Poisson bracket

$$\underbrace{\{f, g\}(x)}_{x \in \mathfrak{g}^*} = x\left([df(x), dg(x)] \right) \in \mathbb{C}, \qquad \mathfrak{g}^{**} = \mathfrak{g}$$

$df(x)$ can be seen as an element of $\mathfrak{g}$.

- Clearly antisymmetric
- Leibniz

$f, g, h \in C^\infty(\mathfrak{g}^*)$

$$\{f, gh\}(x) = x\left([df(x), d(gh)(x)] \right)$$

$$d(gh)(x) = \underbrace{h(x)}_{\in \mathbb{C}} dg(x) + \underbrace{g(x)}_{\in \mathbb{C}} dh(x).$$

$= \dots \rightarrow$ Leibniz identity

Take f, g, h linear functions on $\mathfrak{g}^*$:

$\exists\, u, v, w \in \mathfrak{g}$, $\forall \ell \in \mathfrak{g}^*$

$$f(\ell) = \ell(u)$$
$$g(\ell) = \ell(v)$$
$$h(\ell) = \ell(w).$$

$x \in \mathfrak{g}^*$

Then, $\{f, g\}(x) = x([f, g])$
$$= x([u, v])$$

$$\{\underbrace{\{f, g\}}_{\substack{\text{linear function} \\ \text{on } \mathfrak{g}^* \text{ which} \\ \text{is evaluation at } [u,v]}}, h\}(x) = x([[u,v], w])$$

find π? e_i basis of $\mathfrak{g}$
e_i^* dual basis.

$$f \in \mathcal{C}^\infty(\mathfrak{g}^*), \quad df = \sum \frac{\partial f}{\partial e_i^*} \underbrace{de_i^*}_{\substack{\text{linear fct on } \mathfrak{g}^* \\ // \\ e_i\cdot}}$$

$$\text{so} \quad \{f, g\}(x) = \sum_{i,j} \frac{\partial f}{\partial e_i^*}(x) \frac{\partial g}{\partial e_j^*}(x) \cdot \boxed{x([e_i, e_j])}.$$

$$= df(x) \otimes dg(x)\,(\pi(x)). \qquad \pi = \sum_{i,j} \pi_{ij} \frac{\partial}{\partial e_i^*} \otimes \frac{\partial}{\partial e_j^*}$$

$$= \sum_{i,j} \frac{\partial f}{\partial e_i^*}(x) \frac{\partial g}{\partial e_j^*}(x) \boxed{\pi_{ij}(x)},$$

$$\boxed{\pi_{ij} = \sum_{i,j} \underset{\substack{\Uparrow \\ \mathfrak{g}}}{[e_i, e_j]} \frac{\partial}{\partial e_i^*} \otimes \frac{\partial}{\partial e_j^*}.}$$

5- M, N Poisson.

$M \times N$ has the structure of a Poisson manifold.

$f, g \in \mathcal{C}^\infty(M \times N)$

$$\{f, g\}(x,y) = \{f(-,y), g(\ ,y)\}_M(x) + \{f(x,-), g(x,-)\}_N(y).$$

$(x,y) \in M \times N$

antisymmetry. obvious

Jacobi obvious

Leibniz : use that $(df)(x,y) = d(f(-,y))(x) + d(f(x,-))(y)$

→ do it in local coordinates to convince yourself.

G Lie group. + $m : G \times G \longrightarrow G$ Poisson map.
+ Poisson bracket

$\varphi : (M, \{,\}_M) \longrightarrow (N, \{-,-\}_N)$ Poisson map.

if $\forall\, f, g \in \mathcal{C}^\infty(N)$, $\quad \{f \circ \varphi, g \circ \varphi\}_M = \{f, g\}_N \circ \varphi$.

6- $\Delta : \mathcal{C}^\infty(G) \longrightarrow \mathcal{C}^\infty(G \times G) \overset{\text{dense}}{\supset} \mathcal{C}^\infty(G) \otimes \mathcal{C}^\infty(G)$

$\quad f \longmapsto (x,y) \mapsto f(xy).$

so G Poisson Lie group $\Leftrightarrow \{\Delta f, \Delta g\} = \Delta\{f, g\}$.

7. $m : G \times G \to G$ Poisson map : $\forall f, g \in \mathcal{C}^\infty(G)$ $\quad x_0, y_0 \in G$

$$\{f \circ m, g \circ m\}_{G \times G}(x_0, y_0) = \{f, g\} \circ m (x_0, y_0)$$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \| $

$\qquad \qquad \| \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \{f, g\}(x_0 y_0)$

$\{f \circ m(-, y_0), g \circ m(-, y_0)\}_G(x_0) \qquad \qquad \qquad \qquad \|$

$+ \{f \circ m(x_0, -), g \circ m(x_0, -)\}_G(y_0) \qquad df(x_0 y_0) \otimes dg(x_0 y_0)\,(\pi(x_0 y_0)).$

$\rho_{y_0} : G \longrightarrow G \qquad \lambda_{y_0} : G \longrightarrow G \qquad f \circ m(-, y_0) = f \circ \rho_{y_0}$

$\quad g \longmapsto g y_0 \qquad \qquad g \longmapsto y_0 g.$

$$= d(f\circ \rho_{y_0})(x_0) \otimes d(g\circ \rho_{y_0})(x_0)\,(\pi(x_0))$$
$$+ \; d(f\circ \lambda_{x_0})(y_0) \otimes d(g\circ \lambda_{x_0})(y_0)\,(\pi(y_0)).$$

$\forall f, g.$

$$d(f\circ \rho_{y_0})(x_0) = df(x_0 y_0)\circ d\rho_{y_0}(x_0) \;, \ldots$$

$$\Rightarrow \pi(x_0 y_0) = d\rho_{y_0}(x_0) \otimes d\rho_{y_0}(x_0)\,(\pi(x_0))$$
$$+ \; d\lambda_{x_0}(y_0) \otimes d\lambda_{x_0}(y_0)\,(\pi(y_0)).$$

Remarks: $x_0, y_0 = e \;\rightarrow\; \pi(e) = \pi(e) + \pi(e)$

$$\Rightarrow \pi(e) = 0$$

$x_0^{-1} = y_0 \; : \; \pi(e) = \ldots$

8- No, i is an anti Poisson map: $\{f\circ i, g\circ i\} = -\{f, g\}\circ i.$

9- $S: \mathfrak{g} \rightarrow \mathfrak{g}\otimes\mathfrak{g}.$

13.- $\mathfrak{g}^*$ Poisson Lie group.

$\mathfrak{g}^{**} \simeq \mathfrak{g}$

14-

15- $sl_2(\mathbb{C})$.