TD4-

4.1. $\mathfrak{g}_1, \mathfrak{g}_2$ ss. Lie algebras

$$0 \to \mathfrak{g}_1 \to \mathfrak{g} \xrightarrow{p} \mathfrak{g}_2 \to 0 \qquad \text{splits.}$$

$i \circ f^{-1}$

1- Show that $\mathfrak{g}$ is semisimple.

Take $I \subset \mathfrak{g}$ solvable ideal.

$p(I) \subset \mathfrak{g}_2$ solvable ideal

$\Rightarrow p(I) = (0)$: $I \subset \mathfrak{g}_1$. ideal, solvable.

$\Rightarrow I = (0)$.

So $\mathfrak{g}$ is ss. $\to$ the Killing form is nondegenerate.

$\exists\, \mathfrak{g}'$ s.t. $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}'$

$\mathfrak{g}'$ is an ideal of $\mathfrak{g}$.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "p"] \arrow[d, shift right, "pr_2"'] & \mathfrak{g}_2 \\ \mathfrak{g}' \arrow[u, shift right, "i"'] \arrow[ur, "\sim"] & \end{tikzcd}$$

f ← restriction of p to $\mathfrak{g}'$

- fd ss Lie algebras $\rightsquigarrow$ category $\to$ stable under extensions

$\cap$

fd Lie algebras $\quad f : \mathfrak{g} \to \mathfrak{h}$ morphism of Lie algebras.

Need to show that $\ker f$ is semisimple

$\mathfrak{g} = \mathfrak{g}_1 \times \dots \times \mathfrak{g}_s$ product of simple Lie algebras

$\ker f$ will be semisimple

abelian cat $\ni M, N$

group structure.

$$\underbrace{\mathrm{Ext}^1(N,M)}_{\text{group}} = \left\{ 0 \to M \xrightarrow{\alpha} E \to N \to 0 \right\} / \sim$$

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & E' \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

(equivalence relation).

Central extension : $\mathrm{im}\,\alpha \subset \mathrm{Center}(E)$

ex 4.2.

$$\mathfrak{g} \xrightarrow{ad} \mathrm{Der}(\mathfrak{g}) \hookrightarrow \mathrm{End}(\mathfrak{g})$$
$$x \longmapsto [x,-]$$

$ad(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g})$.
ideal.

$D \in \mathrm{Der}(\mathfrak{g})$ $\quad D' = [D, ad(x)] = [y,-]$
$x \in \mathfrak{g}$ $\quad\quad = ad(y)$ $\quad y = D(x)$.

$z \in \mathfrak{g}$ $\quad D'(z) = D \circ ad(x)(z) - ad(x) \circ D(z)$
$= D[x,z] - [x, D(z)]$
$= [D(x), z] + \cancel{[x, D(z)]} - \cancel{[x, D(z)]}$
$= ad(D(x))(z)$

$\rightarrow$ $D' = ad(D(x))$ $\quad \rightarrow$ $ad(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g})$ is an ideal

ex 4.3

$\mathfrak{g}$ ssimple

V f.d. vector space

$\mathfrak{g} \overset{\rho}{\hookrightarrow} \mathrm{End}(V)$ faithful
$\overset{\psi}{x}$

$V = \mathfrak{g}$
$+ f = ad$.

x ss $\Leftrightarrow$ $f(x)$ is ss
x nilp $\Leftrightarrow$ $f(x)$ is nilpotent

1- $\mathfrak{g}^{ss} \subset \mathfrak{g}$ is $\overset{\text{Zariski}}{\text{open}}$ and dense.

$\mathrm{End}(V)^{ss}$ = diagonalizable endomorphismo is open in End(V) (Zariski), so dense since End(V) is irreducible.

$\rightarrow$ $\mathfrak{g}^{ss} = \rho^{-1}(\mathrm{End}(V)^{ss})$. is open in $\mathfrak{g}$.
not empty : $0 \in \mathfrak{g}$ is semisimple.

$M_n(\mathbb{C}) \supset M_n^{ss}(\mathbb{C})$ is open. $\supset M_n^{rss}$ ← regular semisimple.
How to prove this ?

ss + pairwise ≠ eigenvalues.

$\big\downarrow \chi$

$\mathbb{C}^n$ coefficients of ce polynomial.
$M \in M_n^{rss} \Leftrightarrow \chi_M$ has n distinct root.

"$\chi(M)$ and $\chi(M)'$ have no common roots"

$\Leftrightarrow$ Resultant $(\chi(M), \chi(M)') \neq 0$

"discriminant"

$$\mathrm{Res}(P,Q) \propto \prod_{i,j} (\underbrace{\lambda_i}_{\text{roots of } P} - \underbrace{\mu_j}_{\text{roots of } Q})$$

2- $\mathcal{N}$ is closed and conical.

$$\mathfrak{gl}(V) \xrightarrow[\text{char pol map}]{\chi} \mathbb{C}^n \quad \dim V = n$$

$$\mathcal{N} = \chi^{-1}(0).$$

$$\underbrace{\mathfrak{g} \xrightarrow{\rho} \mathfrak{gl}(V)}_{\text{linear}} \supset \mathcal{N}_V \text{ closed, conical}$$

$\rho^{-1}(\mathcal{N}_V) = \mathcal{N}$, closed conical.

Other properties of the nilpotent cone:

- $\mathcal{N}$ is irreducible.
- $\mathcal{N}$ has a finite number of G orbits if G is simply connected. (G is unique if we require it to be simply connected.)
  a semisimple algebraic Lie $G = \mathfrak{g}$.

not obvious at all.: look for a proof.

$\mathfrak{g} = \mathfrak{sl}_n \to$ Jordan decomp.

$\to$ nilp orbits $\xleftrightarrow{1:1}$ $\mathcal{P}_n$ = partitions of n

$\{\lambda_1 \geq \dots \geq \lambda_k > 0 \mid \sum \lambda_i = n\}$.

$G \curvearrowright X$, $G\cdot x$ locally closed

$$\mathcal{N} = \bigsqcup_{G \text{ orbits}} G \quad \text{locally closed in } \mathcal{N}.$$

$\exists\, G_{reg}$ open in $\mathcal{N}$. $G_{reg} = G\cdot x$

$$G \longrightarrow \mathcal{N} \qquad g \longmapsto g\cdot x.$$

G connected. $\to$ irreducible.

$\mathcal{N} = \overline{G_{reg}}$ regular nilpotent orbit.

For $\mathfrak{sl}_n$:
$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} = N.$$

- For $\mathfrak{sl}_n$: $\dim \mathcal{N} = \dim G_{reg} = \dim SL_n - \dim \mathrm{Stab}_{SL_n}(N)$

$$= n^2 - 1 - (n-1)$$
$$= n(n-1) = 2 \cdot \frac{n(n-1)}{2}$$

g commutes with $N \iff g = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ & \ddots & & \vdots \\ & & \ddots & \lambda_2 \\ 0 & & & \lambda_1 \end{pmatrix}$ $\qquad \dim \mathrm{Stab}_{SL_n}(N) = n-1.$

$g \in SL_n \qquad \lambda_1^n = 1.$

$$\mathrm{Stab}_{SL_n}(N) = \mu_n \times \mathbb{C}^{n-1}$$

- nilcone encodes rep. theoretic properties of SL_n of the Weyl group of SL_n. $= \mathfrak{S}_n$

"Springer theory"

3- $\mathfrak{g}^{ss}$ and $\mathcal{N}$ are $\mathrm{Aut}(\mathfrak{g})$-stable.

$\mathrm{Aut}(\mathfrak{g}) = \{ g \in \mathrm{End}(\mathfrak{g}) \mid g \text{ respects } [-,-] \}$

$a \in \mathrm{Aut}(\mathfrak{g})$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, hook, "\rho"] \arrow[d, "a"'] & \mathfrak{gl}(V) \text{ faithful} \\ \mathfrak{g} \arrow[ur, hook, "\rho"'] & \end{tikzcd}$$

$\rho \circ a$ is a faithful rep. of $\mathfrak{g}$.

$x \in \mathfrak{g}^{ss} \overset{①}{\iff} \rho(x) \in \mathfrak{gl}(V)^{ss}$

$\iff \rho(a(x)) \in \mathfrak{gl}(V)^{ss}$

$\Updownarrow$

$a(x) \in \mathfrak{g}^{ss}.$

Similarly for $\mathcal{N}$.

<u>ex 4.4</u>: Any automorphism of $M_n(\mathbb{C})$ as an associative alg is inner: $M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ for some $A \in GL_n(\mathbb{C})$.

$M \longmapsto AMA^{-1}$

(<u>Skolem-Noether Theorem</u> – applies to any simple central algebra)

A algebra $/ \mathbb{C}$

* central means $Z(A) = \mathbb{C}$
* simple means any two sided ideal in A is 0 or A.

2- Automorphism of $\mathfrak{gl}_n$ not inner?

$$\begin{aligned} \mathfrak{gl}_n &\longrightarrow \mathfrak{gl}_n \\ A &\longmapsto -{}^tA \end{aligned}$$

transpose matrix $({}^tA)_{ij} = A_{ji}$

$$\begin{aligned}[-{}^tA, -{}^tB] &= {}^tA\,{}^tB - {}^tB\,{}^tA \\ &= {}^t(BA) - {}^t(AB) \\ &= -{}^t[A,B].\end{aligned}$$

not inner since inner automorphisms preserve eigenvalues. This one does not

$$\mathrm{Aut}(\mathfrak{sl}_n) = \mathbb{Z}/2\mathbb{Z} \ltimes \overbrace{\mathrm{Ad}(SL_n)}^{\text{adjoint group.}}$$

$\mathrm{Ad} : SL_n \longrightarrow \mathrm{Aut}(\mathfrak{sl}_n)$ conjugation

$\mathrm{Ad}(SL_n) = PSL_n(\mathbb{C}) = PGL_n(\mathbb{C})$.

semi-direct direct product.

[anticipation simple Lie algebras $\overset{1:1}{\longleftrightarrow}$ Dynkin diagrams

$\mathfrak{sl}_n \longleftrightarrow \underbrace{\bullet\!-\!\bullet \cdots\cdots \bullet\!-\!\bullet}_{n-1 \text{ nodes}}$

$\mathfrak{sl}_2 \longleftrightarrow \bullet$

$\mathfrak{sl}_3 \longleftrightarrow \bullet\!-\!\bullet$

In general, $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathrm{ad}(G) \subset \mathrm{Aut}(\mathfrak{g})$

$\mathfrak{g}$ Lie alg

$\{ G \mid \mathrm{Lie}\, G = \mathfrak{g} \}$

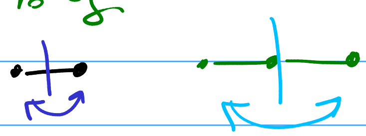
$\mathrm{ad}(G)$ connected "smallest"

biggest $\tilde{G}$ simply connected $\exists$ universal cover.

$\tilde{G} \twoheadrightarrow G \twoheadrightarrow \mathrm{ad}(G)$

$$\mathrm{Aut}(\mathfrak{g}) = \underset{\substack{\| \\ \text{automorphism of the Dynkin diagram associated to } \mathfrak{g}}}{\text{finite group}} \ltimes \mathrm{ad}(G).$$

ex: $\mathfrak{sl}_n$ $n \geq 2$



order 2 auto.

$\mathfrak{g}$ Lie algebra.

4.5 Properties of $U(\mathfrak{g})$.

1- $\forall a, b \in U(\mathfrak{g})$, $ab = 0 \Leftrightarrow a = 0, b = 0$.

$0 = F_{-1} = F_0 \subset F_1 \subset F_2 \cdots \subset U(\mathfrak{g})$ filtration

PBW theorem $\quad gr\, U(\mathfrak{g})$

$$\| $$

$$\bigoplus_{i \geq 0} F_i / F_{i-1} \simeq S(\mathfrak{g})$$

$$= \mathbb{C}[x_i : i \in I]$$

where $\{x_i : i \in I\}$ is a $\mathbb{C}$- basis of $\mathfrak{g}$.

"principal symbol map"

$$\sigma : U(\mathfrak{g}) \longrightarrow gr(U(\mathfrak{g}))$$ (not algebra morphism)

$\sigma(x)$ is the image of x in F_i / F_{i-1} if $x \in F_i \setminus F_{i-1}$.

$\sigma(xy) = \sigma(x)\sigma(y)$ if $\sigma(x)\sigma(y) \neq 0$.

$x \in F_i \setminus F_{i-1}$ $\quad xy \in F_{i+j}$

$y \in F_j \setminus F_{j-1}$

$\sigma(x)\sigma(y) \neq 0 \Rightarrow xy \notin F_{i+j-1}$

so $\sigma(xy) =$ image of xy in F_{i+j} / F_{i+j-1}

$= \sigma(x)\sigma(y)$.

$U(\mathfrak{g})$: $x, y \neq 0$ $\quad \sigma(x) \in gr\, U(\mathfrak{g}) \setminus \{0\}$

$\sigma(y) \in gr\, U(\mathfrak{g}) \setminus \{0\}$

$\sigma(x)\sigma(y) \neq 0$ since $gr\, U(\mathfrak{g})$ is a pol ring so domain.

so $\sigma(xy) = \sigma(x)\sigma(y) \neq 0$ so $xy \neq 0$.

2-a $\quad A$ $\mathbb{C}$-algebra $\quad$ derivation of A

$D \in \mathrm{End}_{\mathbb{C}}(A)$

$D(ab) = aD(b) + D(a)b.$

Derivations of A are certain alg morphisms to $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$

$\cap$ subalgebra

$M_2(A).$

$$A \xrightarrow{\varphi} \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$$

$$a \longmapsto \begin{pmatrix} f(a) & h(a) \\ 0 & g(a) \end{pmatrix}$$

$\varphi(a) = a \cdot 1_{M_2(A)}$

$\Leftrightarrow \quad f(a) = a$

$g(a) = a$

$h(a) = 0$

$\varphi(ab) = \varphi(a)\varphi(b)$

$\Leftrightarrow \begin{cases} f, g \text{ are alg. morphisms } A \to A \\ h \text{ is a twisted derivation of } A. \end{cases}$

$h(ab) = f(a)h(b) + h(a)g(b)$

in particular, if $f = g = \mathrm{id}_A \to h$ is a derivation of A.

b-

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "D \text{ derivation}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{g} \arrow[d] \\ U(\mathfrak{g}) \arrow[r, dashed, "\tilde{D} \text{ lift}"'] & U(\mathfrak{g}). \end{tikzcd}$$

associative

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "D'"] \arrow[d, bend right] & \begin{pmatrix} U(\mathfrak{g}) & U(\mathfrak{g}) \\ 0 & U(\mathfrak{g}) \end{pmatrix} \\ U(\mathfrak{g}) \arrow[ur, dashed, "\tilde{D}'"'] \end{tikzcd}$$

$$x \longmapsto \begin{pmatrix} x & D(x) \\ 0 & x \end{pmatrix}$$

$\circlearrowleft$

D' is a Lie algebra morphism.

$\exists!$ by universal prop of universal env. algebra.

$\tilde{D} : U(\mathfrak{g}) \longrightarrow \begin{pmatrix} U(\mathfrak{g}) & U(\mathfrak{g}) \\ 0 & U(\mathfrak{g}) \end{pmatrix}$ algebra morphism

If $x \in \mathfrak{g} \subset U(\mathfrak{g})$

$$\tilde{D}'(x) = D'(x) = \begin{pmatrix} x & D(x) \\ 0 & x \end{pmatrix}.$$

$\tilde{D}'$ alg morphism $\Rightarrow \forall x \in U(\mathfrak{g})$, $\tilde{D}'(x) = \begin{pmatrix} x & \tilde{D}(x) \\ 0 & x \end{pmatrix}$.

By a., $\tilde{D}$ is a derivation of $U(\mathfrak{g})$.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "D"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{g} \arrow[d] \\ U(\mathfrak{g}) \arrow[r, "\tilde{D}"'] & U(\mathfrak{g}) \end{tikzcd}$$

$\tilde{D}$ unique.

$D = [x, -]$ for $x \in \mathfrak{g}$, is a derivation of $\mathfrak{g}$.

$\rightsquigarrow \tilde{D}$

$a, b \in \mathfrak{g}$

$$\begin{aligned}\tilde{D}(ab) &= a\tilde{D}(b) + \tilde{D}(a)b \\ &= a[x,b] + [x,a]b \\ &= a(xb - bx) + (xa - ax)b \\ &= xab - abx \\ &= [x, ab].\end{aligned}$$

More generally, $\forall y \in U(\mathfrak{g})$, $\tilde{D}(y) = [x, y]$.

3- $U(\mathfrak{g}) \simeq U(\mathfrak{g})^{op}$

If $(A, *)$ is an associative algebra, $(A^{op}, *_{op})$ is the algebra with the opposite product:

$$a *_{op} b = b * a.$$

$\mathfrak{g}$ Lie alg $\quad \mathfrak{g}^{op} \ni x, y \quad [x,y]_{op} = -[x,y] = [y,x].$

$$\begin{array}{ccc} \mathfrak{g} \longrightarrow U(\mathfrak{g}) & x \in \mathfrak{g} \\ \downarrow S & \downarrow \\ U(\mathfrak{g})^{op} & -x. \\ \| & \\ U(\mathfrak{g}^{op}) & \end{array}$$

$T\mathfrak{g} / \langle\langle x \otimes y - y \otimes x - [x,y] \quad x, y \in \mathfrak{g} \rangle\rangle.$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\checkmark"] \arrow[d] & \mathfrak{g}^{op} \arrow[d] \\ U(\mathfrak{g}) \arrow[r, dashed, leftrightarrow, "\exists"'] & U(\mathfrak{g}^{op}) \simeq U(\mathfrak{g})^{op} \end{tikzcd}$$

$x \longmapsto -x$

5. A associative algebra

$0 = F_{-1} \subset F_0 \subset F_1 \subset \ldots \qquad F_i F_j \subset F_{i+j}.$

$$\bigcup_{i \geq 0} F_i = A$$

If $\mathrm{gr}\, A$ is noetherian then A is noetherian.

ex 4.7. 1. $K(x,y) = 2n\, \mathrm{Tr}(xy)$

For $\mathfrak{sl}_2$: $K(x,y) = 4\,\mathrm{Tr}(xy)$

2. $\mathfrak{sl}_2 = \langle e, f, h \rangle \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (nice) basis of $\mathfrak{sl}_2$

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\frac{f}{4} \quad \frac{e}{4} \quad h/8$

(e^*, f^*, h^*) dual basis

$C = ee^* + ff^* + hh^*$

$= \frac{ef}{4} + \frac{fe}{4} + \frac{h^2}{8}$

$K(h,h) = 8$

$K(h, \frac{h}{8}) = 1$

$K(\frac{e}{f}, \frac{h}{8}) = 0$

$$\kappa(e,f) = 4\cdot \operatorname{Tr}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = 4$$

$$= \frac{1}{4}\left(ef + fe + \frac{h^2}{2}\right)$$

$C \in U(\mathfrak{g})$.

$\in Z(\mathfrak{g})$. For $\mathfrak{sl}_2$, $Z(U(\mathfrak{sl}_2)) = \mathbb{C}\cdot\left(ef + fe + \frac{h^2}{2}\right)$

$Z(\mathfrak{sl}_2) = (0)$

3 - $\mathfrak{sl}_n$ E_{ij} $i \neq j$
$E_{ii} - E_{i+1,i+1}$ $1 \leq i \leq n-1$.

4.9: $\kappa \in \operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}\otimes\mathfrak{g}, \mathbb{C}) \overset{\text{as } \mathfrak{g}\text{-reps.}}{\simeq} \operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathfrak{g}^*)$

$\mathfrak{g} \circlearrowleft$ $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is simple

$\searrow \mathfrak{gl}(\mathfrak{g}^*)$ is also simple.

$\Rightarrow \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}^*) =$ morphisms of repr $\mathfrak{g} \to \mathfrak{g}^*$.
$= \mathbb{C}$

the Killing form for simple Lie algebras give $\boxed{\mathfrak{g} \simeq \mathfrak{g}^*.}$

Poisson variety.

$f, g \in \mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}^{**})$
$= S\mathfrak{g}$

V vector space
$\mathbb{C}[V] = S(V^*)$

$\{f, g\} = [f, g]$ if $f, g \in S^1\mathfrak{g} = \mathfrak{g}$.

$\{f_1 \cdots f_n$
$f_1 \in \mathfrak{g}$, $f_n \in \mathfrak{g}$