G alg group

$k[G]$

TD 3

Lie bracket $[d,d'] = (d\otimes d')\otimes\Delta - (d'\otimes d)\otimes\Delta$

$$\mathfrak{g} = \{ d : k[G] \to k \mid d(fg) = df\, g(e) + \underbrace{f(e)}_{\in k} dg \}$$

$$\| $$

$$\Gamma(G, TG)^G = \{ \delta : k[G] \to k[G] \text{ derivation, left invariant } (id\otimes\delta)\circ\Delta = \Delta\circ\delta \}$$

$\subset \operatorname{End}_k(k[G])$ induced lie bracket

Last time.

Check that preserve the lie bracket.

ex 3.4 · $\mathfrak{sl}_2 \subset \mathfrak{gl}_2 = 2\times 2$ $\mathbb{C}$-matrices.

$\|$ traceless 2×2 matrices.

$$\mathfrak{sl}_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$$

1- · $\mathfrak{sl}_2$ is a simple lie algebra.

· no nontrivial ideal.

$0 \neq \imath \subset \mathfrak{sl}_2$ ideal

$x = ae + bf + ch \in \mathfrak{sl}_2$, $(a,b,c) \neq (0,0,0)$.

$[e,x]$, $[h,x]$ . . . Check that $\imath = \mathfrak{sl}_2$.

if $x = e$, · $[e,f] = h \in \imath$

$[h,f] = \underline{-2}\, f \in \imath$

In particular, $\mathfrak{sl}_2$ not simple in characteristic 2 : $(e) \subset \mathfrak{sl}_2$ non trivial ideal.

2- V f.d irr rep. $h \circlearrowright V \rightsquigarrow V_\lambda$ eigenspace. $\lambda \in \mathbb{C}$

$$e(V_\lambda) \subset V_{\lambda+2} \quad , \quad f(V_\lambda) \subset V_{\lambda-2}.$$

$v \in V_\lambda$. $hv = \lambda v$. $h\, ev = \underbrace{[h,e]}_{=2e} v + \underbrace{eh v}_{\lambda ev}$

$= (\lambda+2) e\cdot v$

$v \in V_\lambda,\ v \neq 0,\qquad \underset{\lambda}{ev},\ \underset{\lambda+2}{e^2v}, \dots, \underset{\lambda+2n}{e^nv}\qquad V$ f.d $\Rightarrow$ for $n \gg 0$, $e^n \cdot v = 0$

In the same way, for $m \gg 0$, $f^m \cdot v = 0$.

3- Take $v \in V$ st. $e \cdot v = 0$. We can assume $v \in V_\lambda$ (say)

$$V_\lambda \ni e^n \underbrace{f^n v}_{V_{\lambda-2n}} \in \mathbb{C} v.$$

$$efv = \underset{\overset{\shortparallel}{h}}{[e,f]} v + \cancel{fe v}^{\,0}$$

$$= hv = \lambda v.$$

$$e^2f^2v \overset{?}{=}$$

$$\underset{\overset{\shortparallel}{h}}{(ef-fe)} \cdot fv = ef^2v - f\underbrace{efv}_{\lambda v}$$

$$\shortparallel$$

$$(\lambda-2)fv$$

$$\rightarrow (\lambda-2)fv = \boxed{ef^2v} - \lambda f v.$$

ex

$$(\lambda-2)\, efv = e^2f^2v - \lambda efv$$

$$\lambda(\lambda-2)v = e^2f^2 v - \lambda^2 v.$$

$$\Rightarrow e^2f^2v = \lambda(\lambda-1)v.$$

In fact $e^nf^nv = \lambda(\lambda-1)\cdots(\lambda-(n-1))v$. (induction)
$\forall n \geq 1$.

take n (minimal) st $f^nv = 0 \rightarrow \exists\, 0 \leq k \leq n-1$ s.t. $\lambda = k$.
$\Rightarrow \lambda \in \mathbb{N}$.

4. $\mathrm{Tr}(h \mid V) = 0$. $\quad h = [e,f]. \rightsquigarrow \rho : sl_2 \longrightarrow gl(V)$

h acts by $[\rho(e), \rho(f)]$

so trace $(h \mid V) = 0$.

4. V irreducible.

$v \in V_\lambda$. $e \cdot v = 0$

$\mathfrak{sl}_2 \cdot v \subseteq V$ since V irreducible

$\quad \| \quad$ subrep of V

$$\langle v, fv, f^2v, \dots, f^n v, \dots \rangle$$

$e^n f^n \in \mathbb{C}v$ — Take n minimal s.t. $f^n v = 0$, so

$ef^k v \in \mathbb{C} f^{k-1} v$. — that $f^{n-1} v \neq 0$

and $\{v, fv, \dots, f^{n-1} v\}$ is a $\mathbb{C}$-basis of V.

$$\mathrm{Tr}(h \mid V) = \lambda + \lambda - 2 + \dots + \lambda - (n-1)\cdot 2$$
$$= n\lambda - 2 \cdot \frac{n(n-1)}{2}$$

$$\Rightarrow \lambda = n-1 = \dim V - 1.$$

5. If V irr, n-dim repr of $\mathfrak{sl}_2$,

$$V = \bigoplus_{k=0}^{n-1} f^k v, \qquad hv = (n-1)v$$

highest weight vector

$$\begin{array}{ll} (n-1) & v \\ \lambda v & fv \\ (\lambda-1)\times & f^2 v \\ & \vdots \\ -(n-1) & f^{n-1} v \end{array}$$

$$ef^2 v = (\lambda - 1) fv$$

Classification of (fd) reps of $\mathfrak{sl}_2$: For any $n \geq 0$, there is a unique iso class of rep of dim n given as above.

of Lie algebra, V, W 2 reps

exercise 3.5: $V \otimes W$ is a $\mathfrak{g}$ rep.

$x \in \mathfrak{g}$

$$x(v \otimes w) = xv \otimes w + v \otimes xw.$$ tensor structure on Mod $\mathfrak{g}$.

[comes from coalgebra structure on $U(\mathfrak{g})$ = enveloping algebra]

If A is a bialgebra, A-Mod is a tensorcategory

$M, N \in A\text{-Mod}$, $A \curvearrowright M \otimes N$

$a \in A$ $\Delta(a) \in A \otimes_{\mathbb{C}} A \curvearrowright M \otimes N$.

$x \in \mathfrak{g}$

associative algebra $\xrightarrow{}$ $U(\mathfrak{g}) \hookleftarrow \mathfrak{g}$ $\Delta(x) = x \otimes 1 + 1 \otimes x$.

$\mathfrak{g}$ generated $U(\mathfrak{g})$ as an associative algebra].

sl_2 simple V, W irr reps of sl_2

How do $V \otimes W$ decompose into irreducible summands?

1. $$ch(V) = \sum_{n \in \mathbb{Z}} \dim V[n]\, t^n \in \mathbb{Z}[t, t^{-1}]$$

$n \in \mathbb{Z}$ $V[n]$ = n-th weight space $=$ n-eigenvalue space of h.

$\mathbb{C}h \subset sl_2$ "Cartan subalgebra"

$$ch(V \oplus W) = ch(V) + ch(W).$$

$$ch(V \otimes W) \overset{?}{=} ch(V)\,ch(W).$$

Take $v \in V[n]$, $w \in V[m]$,

$$\begin{aligned} h(v \otimes w) &= hv \otimes w + v \otimes hw \\ &= nv \otimes w + v \otimes mw \\ &= (n+m)\, v \otimes w. \end{aligned}$$

so $$V \otimes W = \bigoplus_{n, m \in \mathbb{Z}} \underbrace{(V[n] \otimes V[m])}_{\subset\, (V \otimes W)[m+n]}.$$

$k \in \mathbb{Z}$

$$(V \otimes W)[k] = \bigoplus_{n+m=k} V[n] \otimes V[m].$$

2- $V \simeq W \Leftrightarrow ch(V) = ch(W) \overset{t=1}{\Rightarrow} \dim V = \dim W$

$\Rightarrow$ obvious

$\Leftarrow$ less obvious: induction on $\dim V = \dim W$.

× If $\dim V$, $V = (0)$ — ok.

* Assume $\Leftarrow$ is true for $\dim \leq n-1$. Assume $\dim V = n$.

$$ch(V) = \sum_{n \in \mathbb{Z}} \dim V[n]\, t^n$$

n max s.t. $V[n] \neq 0$. $W[n] \neq 0$

$v \in V[n]$.

$\langle v \rangle \subset V$ $\quad w \in W[n]$

$\langle w \rangle \subset W$

$\underbrace{\text{isotyp}}_{\text{irr}}$ $\overbrace{n+1 \dim \text{ repr of } sl_2}^{\text{irr}}$ $\quad$ $\overset{\text{SI}}{(n+1)}$ dim irr rep of sl_2.

$V = \langle v \rangle \oplus V'$ $\quad$ $W = \langle w \rangle \oplus W'$

$$ch(V') = ch(V) - ch(\langle v \rangle) = ch(W').$$

By induction hypothesis, $V' \simeq W' \Rightarrow V \simeq W$.

3- V_m = m-dimensional $\overset{\text{irr}}{\text{repr}}$ of sl_2. Assume $m \leq n$

$\min(m,n) = m$

$$V_m \otimes V_n \simeq \bigoplus_{i=1} V_{m+n-2i}.$$

$$V_m: \begin{array}{cc} m-1 & v \\ \vdots & \vdots \\ -(m-1) & f^{m-1} v \end{array}$$

$$ch(V_m) = \sum_{j=0}^{m-1} t^{-(m-1)+2j} = \frac{1}{t^{m-1}} \frac{1-t^{2m}}{1-t^2}.$$

$$ch(V_n) = \frac{1}{t^{n-1}} \frac{1-t^{2n}}{1-t^2}$$

$$\sum_{i=1}^{m} ch(V_{m+n-2i}) = \sum_{i=1}^{m} \frac{1}{t^{m+n-2i-1}} \frac{1-t^{2(m+n-2i)}}{1-t^2}$$

$\to$ Calculation.

4. $V_1 = 2$-dim irr repr of sl_2, defining rep of sl_2.

$$\begin{matrix} sl_2 \\ \cap \\ gl_2 \end{matrix} \curvearrowright \mathbb{C}^2 = \langle \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{v}, \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{fv} \rangle \qquad e = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$e v = 0$

$f v = \begin{pmatrix}0\\1\end{pmatrix}$

$\underbrace{V_1^{\otimes 2}}_{4\text{-dim}} \supset \quad V_{2+2-2} = V_1 = 2$-dim

$V_2 + 2 - 4 = V_0 = 0$-dim

3- V_m $\quad m+1$ - dim rep of sl_2.

$$V_1^{\otimes 2} = V_{1+1-1} \oplus V_{1+1-2}$$
$$= \underbrace{V_1}_{2\text{-dim}} \oplus \underbrace{V_0}_{1\text{-dim}}$$

exercise Correct question 3 and 4.

$m \leq n$

! $V_m \otimes V_n = \bigoplus_{i=0}^{m} V_{m+n-2i}$

V_m $\quad m+1$-dim irr rep of sl_2

& begin at $i=0$.

$V_1 \otimes V_1 = \underbrace{V_2}_{3\text{-dimensional}} \oplus \overbrace{V_0}^{1\text{-dimensional}}$

$V_1^{\otimes n} \supset V_n$.

<u>ex 3.10</u>. 1- $\mathrm{Trace}\,(\underbrace{ad_x \circ ad_y}_{sl_n}) = 2n \cdot \mathrm{Trace}\,(xy)$ $\quad x, y \in sl_n$, $\quad xy \in gl_n$

sl_2 ... <u>calculation</u>

2- sl_n is a simple Lie algebra.
semi simple : the Killing form is non degenerate :
If $\underset{\sim}{tr(xy)} = 0$ $\forall y \in sl_n$, then $x = 0$.

E_{ij} $i \neq j$ ...
$\uparrow$
sl_n

3- $sl_n \ni \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$ $\begin{cases} E_{ij}, & i \neq j. \\ E_{ii} - E_{jj} & i \neq j \end{cases}$

Lie algebra gens : $e_i = E_{i,i+1}$ $\quad 1 \leq i \leq n-1$
$f_i = E_{i+1,i}$ $\quad 1 \leq i \leq n-1.$
$h_i = E_{ii} - E_{i+1,i+1}.$

$e_i, f_i, h_i \rightsquigarrow$ $\begin{pmatrix} 0 & & \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \\ & & 0 \end{pmatrix}$ $a, b, c, d \in sl_2$.
gives $sl_2 \hookrightarrow sl_n$.

$[e_i, f_i] = h_i$, $[h_i, e_i] = 2e_i$, $[h_i, f_i] = -2f_i$.

if $|i-j| \geq 2$

$$\begin{array}{c} i \\ i+1 \\ \\ j \\ j+1 \end{array} \begin{pmatrix} \boxed{\times} & \\ & \boxed{\times} \end{pmatrix} \text{ if } \quad j \geq i+2.$$

if $|i-j| \geq 2$, $[e_i, f_j] = 0 = [e_i, h_j] = [e_i, e_j] = [f_i, f_j] = \ldots$

$[h_i, h_j] = 0 \quad \forall i, j$

$[e_i, e_j] = 0$

$[f_i, f_j] = 0.$ $\qquad [e_i, \cancel{e_{i+1}}] =$
$[e_i, f_{i+1}] =$

$\rightsquigarrow$ Cartan matrix of sl_n : $(n-1) \times (n-1)$ matrix $\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} = (a_{ij})_{i,j}$

Relations

$$[e_i, f_i] = h_i \qquad [h_i, e_j] = a_{ij} e_j$$
$$[h_i, e_i] = 2e_i \qquad [h_i, f_j] = -a_{ij} e_j$$
$$[h_i, f_i] = -2f_i$$

part of

Serre relations $i \neq j$: $\operatorname{ad}(e_i)^{1-a_{ij}}(e_j) = 0$

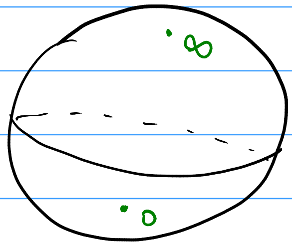
$$\Downarrow$$

$(e_i, e_j) = 0$ if $|i-j| \geq 2$.

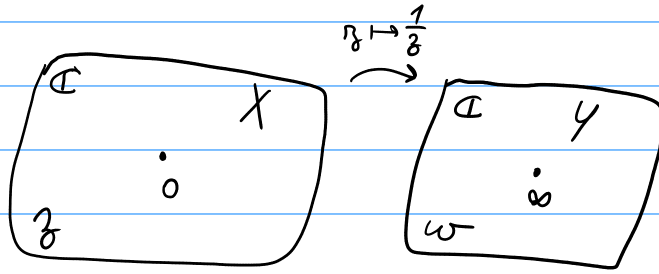
$$\operatorname{ad}(f_i)^{1-a_{ij}}(f_j) = 0.$$

$\rightsquigarrow$ not completely immediate to show these are all the relations.

ex 3.11 $\quad \mathbb{P}^1_{\mathbb{C}}$



polynomial vector fields.



A polynomial vector field on $\mathbb{C}$ is $P(z)\dfrac{\partial}{\partial z}$

A pol v. f on $\mathbb{P}^1_{\mathbb{C}}$ is $P(z)\dfrac{\partial}{\partial z}$ on $\mathbb{C}$, and has to extend at ∞.

$P(z)\dfrac{\partial}{\partial z} = X$ vector field on $\mathbb{C}$

$$f = z \mapsto \frac{1}{z}.$$

$$Y\left(\frac{1}{z}\right) = df(z) \cdot X(z).$$
$$= -\frac{1}{z^2} P(z) \frac{\partial}{\partial w}$$

has to be polynomial.
$\Rightarrow$ degree $P \leq 2$.

$$Y(w) = -w^2 P\left(\frac{1}{w^2}\right)\frac{\partial}{\partial w}$$

$w = \frac{1}{z}$

Poly. vectors fields on $\mathbb{P}^1$ $= \left\langle \overset{f}{\dfrac{\partial}{\partial z}}, \overset{h}{2z\dfrac{\partial}{\partial z}}, \overset{e}{z^2\dfrac{\partial}{\partial z}} \right\rangle_{\mathbb{C}\text{-vs.}}$

$$\left[\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}\right] = 2z\frac{\partial}{\partial z}; \quad \left[2z\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right] = -2\frac{\partial}{\partial z}$$

$$\left[2z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}\right] = 2z\frac{\partial}{\partial z}z^2\frac{\partial}{\partial z} - z^2\frac{\partial}{\partial z}2z\frac{\partial}{\partial z}$$

$$= 2z\cdot 2z + \cancel{2z z^2\frac{\partial^2}{\partial z^2}} - z^2\cdot 2\frac{\partial}{\partial z} - \cancel{z^2 2z\frac{\partial^2}{\partial z^2}}$$

$$= 2z^2\frac{\partial}{\partial z}$$

polynomial vector fields on $\mathbb{P}^1 \simeq \mathfrak{sl}_2$.

$\mathbb{P}^1$ is the flag variety of $\mathfrak{sl}_2$.

$$\mathcal{F}(\mathfrak{sl}_n) = \{(0 \subseteq V_1 \subseteq \dots \subseteq V_n = \mathbb{C}^n) \mid \dim V_i/V_{i-1} = 1 \quad 1\leq i\leq n\}.$$

flag variety of $\mathfrak{sl}_n$.

$\mathcal{F}(\mathfrak{sl}_2) \simeq \mathbb{P}^1$.

In general, polynomials vector fields on the flag variety $\mathcal{F}(\mathfrak{sl}_n)$, you get $\mathfrak{sl}_n$.

If $\mathfrak{g}$ is a reductive Lie algebra, the flag variety of $\mathfrak{g}$ is the variety of Borel subalgebra of $\mathfrak{g}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ‖

maximal, solvable Lie algebra.

Borel subalgebra.

$\text{Lie } B \subset \text{Lie } G = \mathfrak{g}$

$\underline{G = GL_n}$

G reductive connected.

$\mathfrak{g} = \text{Lie}(G)$, $\quad B \subset G$ Borel subgroup.

closed, solvable, max.

flag variety of $\mathfrak{g} \simeq G/B$.

- all Borel subalgebras of $\mathfrak{g}$ are conjugated to each other
- the stabiliser of a Borel subalgebra is a Borel subgroup.

$\rightsquigarrow$ important to construct in a geometric way reps of Lie algebras.

<u>ex 3.3:</u> Find an exact sequence $0 \to \underset{\text{nilp.}}{i} \to \underset{\text{not nilp.}}{\mathfrak{g}} \to \underset{\text{nilp}}{\mathfrak{g}/i} \to 0$

2-dimension, nontrivial Lie algebra:

$$\mathfrak{g} = \mathbb{C}x \oplus \mathbb{C}y \quad , [x,y] = y .$$

$$0 \to \mathbb{C}y \to \mathfrak{g} \to \mathbb{C}x \to 0 ,$$

$\mathfrak{g}$ is not nilpotent. $\quad [\mathfrak{g},\mathfrak{g}] = \mathbb{C}y = C^1$

$$C^2 = [\mathfrak{g}, C^1] = \mathbb{C}y$$

don't reach 0.

$$\mathfrak{g} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$ is nilpotent

$$\mathfrak{g} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \qquad 0 \to \underset{\text{nilp.}}{\mathfrak{u}} \to \mathfrak{b} \to \underset{\text{diag}}{\mathfrak{t}} \to 0$$

$\mathfrak{b}$ is not nilpotent

solvable.