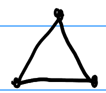
TD2

2.1 - Lie Kolchin: G solvable, connected alg group $\Rightarrow$ every f.dim representation of G is triangular.

- When G is not connected?
- A finite group is an algebraic group.
- $D_{2n} \leftarrow$
- $\mathfrak{S}_3 = \{\text{bijections } \{1,2,3\} \to \{1,2,3\}\}$
- $D\mathfrak{S}_3 \subset \underbrace{\mathcal{A}_3}_{\text{even permutations}} = \{e, (1,2,3), (1\,3\,2)\}$ commutative $\Rightarrow D^2\mathfrak{S}_3 = \{e\}$.

$x, y \in \mathfrak{S}_3 \Rightarrow \cancel{x}\, y x^{-1} y^{-1}$ has signature 1 so $\mathfrak{S}_3$ is solvable.

Our 2-dim repr of $\mathfrak{S}_3$: isometries of triangle.



$\Rightarrow$ not triangular.

ex 2.2. X quasi- ~~projective~~ $\to$ X is open in a projective variety
affine $\to$ X is open in an affine variety

2.) $\mathbb{G}_a \curvearrowright \mathbb{P}^1$.

$$\mathbb{G}_a \hookrightarrow GL_2 \quad \text{closed subgroup}$$
$$x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

$k = \bar{k}$

$GL_2(k) \curvearrowright \mathbb{P}^1(k) \ni [x:y]$ $\quad g \cdot [x,y] = [ax+by : cx+dy]$.
$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Aside [over any base scheme S, $GL_{2/S} \curvearrowright \mathbb{P}^1_S$.

T scheme over S. $\quad \mathbb{P}^1_S(T) = \{\mathcal{L}$ invertible $\mathcal{O}_T$-module with two sections which generate $\mathcal{L}$, s_1, s_2. $\}$]

$\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}\cdot[x:y]=[x+\alpha y:y]$

if $y=0$: $[x:y]=[1:0]$ is fixed, its orbit is closed.

if $y\neq 0$, $\to y=1$ $\{[x+\alpha:1]:\alpha\in k\}=\underset{\text{open in } \mathbb{P}^1_k.}{\overset{\simeq \mathbb{A}^1_k}{\mathbb{P}^1_k\setminus\{[1:0]\}}}$.

1) X quasi-affine. For simplicity, take X affine.

$X\hookrightarrow Y$ Y affine variety $\Rightarrow$ take $X=Y$

$G \circlearrowright X$, G unipotent; $Y = \operatorname{Spec} A$. A is f type, k-algebra reduced,

$O=G\cdot x$ orbit for $x\in X$

$O\subset\overline{O}$, O is open in $\overline{O}$.

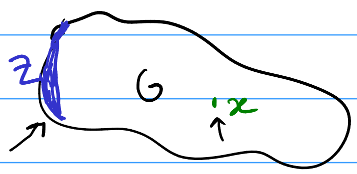
[$\overline{O}$ G-stable "union of G-orbits]

$Z:=\overline{O}\setminus O\subset O$, closed subvariety of $\overline{O}$, of X

$I\subset A$ defining Z. Z is G-invariant.

I is stable under G (regular represent. $G\times A\longrightarrow A$, $(g,f)\longmapsto f(g^{-1}-)$.)

Take a point $x\in O$ $\to$ By (NS), can find a function $f\in A$, $f\in I$ ($f|_Z=0$) and $f(x)=1$.



tells you a closed subset of X is determined by the fcts vanishing on it

$Z\subset Z\cup\{x\}\subset X$ closed

$\to \exists f\in I$, $f\neq 0$ non trivial

$G\times I\to I$ restrict of reg rep. I: k-vector space

Course: $A=\bigcup_{i\in I}V_i$ increasing union of G-stable f dim subspaces of A.

$I=\bigcup_{i\in I}V_i\cap I$

$\exists i\in I$, $f\in I$.

$\exists$ f.dim repr $\overset{A}{\overset{\cup}{V}}$ of G. $V \neq 0$

$\overset{V}{\underset{f}{\overset{\cup}{\,}}}$ $\quad\quad$ $\overset{I}{\overset{\cup}{V}}$

G unipotent $\Rightarrow$ every repr of G has fixed vector, $\underset{\psi}{h} \neq 0$

$g \cdot h = h \ \forall g \in G$ so h has to be constant on G.
But $G \subset \overline{G}$ is dense so h is constant on $\overline{G}$ and
h vanishes on Z $\Longrightarrow$ h vanishes on $\overline{G}$

For this to be correct, replace X by $\overline{G}$.

X affine.
$G \curvearrowright$.

G orbit. $X = \overline{G} \supset G$
$Z = \overline{G} \setminus G$. closed.

if Z not empty, $\underset{f}{\overset{\psi}{I}}$ ideal of Z is not (0). I G-stable
$\underset{A = k[X]}{\cap}$.

$\left\{\begin{matrix} \text{fcts vanishing} \\ \text{on } Z \end{matrix}\right\} = I = \bigcup W_i$ increasing, f.dim.
$\quad\quad\quad\quad G \curvearrowright$

$\exists i \quad W_i \neq 0.$ $\neq 0$.
$\exists h \in W_i$, G-stable.
h vanishes on Z.
h is constant on $\overline{G} = X$. $\Big\} \Rightarrow h = 0$. contradiction

ex 2.4: $\underset{\cup}{G}$ solvable alg group $(D^n G = \{e\}$ for $n >> 0)$.
$\quad\quad U$
$U = \{$unipotent elements$\}$ closed alg subgroup.

$\exists \quad G \hookrightarrow GL_m$

$f \searrow T_m$ triang matrices $\nearrow$

$U_m \subset T_m$ unipotent, solvable

$U = f^{-1}\left(\underbrace{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{U_m}\right)$

obvious that U_m is connected.

$f(G) \cap U_m$.
$\quad //$

$U_m \cong \mathbb{A}^{m(m-1)/2}$. connected.

less obvious than U is connected.

But if $g \in U$. $\overline{g^{\mathbb{Z}}} = \overline{\{g^n : n \in \mathbb{Z}\}} \subseteq \mathbb{G}_a$ subgroup of U.

$i : \mathbb{G}_a \hookrightarrow U$ closed immersion $\quad$ $i(\mathbb{G}_a)$ connected containing g and e

$\Rightarrow U$ is connected.

ex 2.3: Chevalley's thm → give a structure of quasi-projective variety on G/H.

$G \rightsquigarrow G(k)$, $H \rightsquigarrow H(k)$, $H(k) \subset G(k)$ abstract groups → $G(k)/H(k)$ - set theoretic.

Find Y alg quasi variety st. $Y(k) = G(k)/H(k)$

$G \hookrightarrow GL(V)$ $\quad$ $H = \mathrm{Stab}_D(G)$ $\quad$ $D \subset V$

$\rightsquigarrow$ $G(k)/H(k) \xrightarrow{\sim} G(k) \cdot D \subset \mathbb{P}(V)$

set theoretical orbit → locally closed.

$(G \cdot D)(k) = k$ points of the orbit of D,

qp variety.

Prove Chevalley's thm

Recall: to construct a closed immersion $G \to GL_n$ for some n,

we took $V \subset k[G]$

subspace, V generated $k[G]$ as a k-algebra and V finite dimensional

V is G-invariant.

For Chevalley thm, let $I \subset k[G]$ be the ideal of functions vanishing on H.

Choose V as above with the additional assumption that V contains generators of $I = (f_1, \ldots, f_d)$.

Get $G \underset{\text{closed.}}{\hookrightarrow} GL(V)$ $\quad$ G acts on V.

$W \subset V$ $\quad$ $W = \{ f \in V \mid f|_H = 0 \}$ subspace of V.

$\; \| $

$V \cap I$

Show that $\{ g \in G \mid gW \subset W \} = H$.

$\supset$ if $h \in H$, and $f \in W$, $h \cdot f = f(h^{-1} -)$ vanishes on H so $H \subset \{ g \in G \mid gW \subset W \}$.

$\subset$ use that W contains generators of I.

If $g \in G$ s.t. $gW \subset W$, $\forall f \in W$, $g \cdot f|_H = 0$

$\Downarrow$ apply to e

$f(g^{-1}) = 0$.

$\Rightarrow \forall f \in I,\ f(g^{-1}) = 0$ since W generated I as an ideal.

$\Rightarrow g^{-1} \in H$

$\Rightarrow g \in H$ since H is a group.

So $H = \{ g \in G \mid gW \subset W \}$. $\quad$ $\dim W = \omega$

$G \hookrightarrow GL(V) \rightsquigarrow G \curvearrowright Grass(\omega, V) \ni W$

$H = Stab_W(G)$.

· Plücker embedding : $Grass(\omega, V) \hookrightarrow \mathbb{P}^N$ $\quad$ $N = \binom{\dim V}{\omega} - 1$.

Afferent.

$G \hookrightarrow GL(V)$

$G \curvearrowright V \rightsquigarrow G \curvearrowright \Lambda^\omega V \supset \Lambda^\omega W$ line

$\rightsquigarrow G \to GL(\Lambda^\omega V)$

closed immersion $\leftarrow$ Take V s.t. $W \subsetneq V$ have generated $k[G]$, $W \subset I$

$\rightarrow$ prove that if $g \in G$, g acts trivially on $\Lambda^\omega V \Rightarrow g \in e$.

$\hookrightarrow$ exercise of linear algebra.

$N = \dim V$

$$V = \mathrm{Vect}(v_1, \dots, v_N)$$
$$\Lambda^\omega V = \mathrm{Vect}(v_{i_1} \wedge \dots \wedge v_{i_\omega} \,:\, i_1 < i_2 < \dots < i_\omega).$$

$$g(v_{i_1} \wedge \dots \wedge v_{i_\omega}) = (g v_{i_1}) \wedge \dots \wedge (g v_{i_\omega}) = v_{i_1} \wedge \dots \wedge v_{i_\omega}$$
$\forall\, v_{i_1}, \dots, v_{i_\omega}$

$\Rightarrow g = e.$

<u>exercise</u>.

<u>ex 2.5</u> $\Rightarrow$ eg. induction on $\dim V$.

2- <u>Lie algebras</u>

<u>ex 2.6</u> G, $\Gamma(G, TG) = \mathrm{Der}(k[G])$

$\delta : k[G] \longrightarrow k[G]$

Leibniz rule.

$\delta(fg) = (\delta f) g + f(\delta g).$

and $\Gamma(G, TG)^G$ left invariant derivations.

$g \in G$. $\quad \overset{\circlearrowright G}{k[G]} \xrightarrow{\delta} \overset{\circlearrowright G}{k[G]}$ regular rep $\qquad \delta$ commutes with reg. repr.

$(k[G], \Delta, \varepsilon, -)$ Hopf algebra

$f \longmapsto ((g, g') \mapsto f(gg'))$

δ is left invariant if

$$\begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[d, "\delta"] \arrow[dr, phantom, "\circlearrowleft"] & k[G] \otimes k[G] \arrow[d, "\mathrm{id} \otimes \delta"] \\ k[G] \arrow[r, "\Delta"] & k[G] \otimes k[G] \end{tikzcd}$$

$f \longmapsto (\delta f)$

$(g, g') \mapsto (\delta f(g -))(g')$

commutes:

$$\boxed{(\mathrm{id} \otimes \delta) \circ \Delta = \Delta \circ \delta}$$

$(g, g') \mapsto (\delta f)(gg')$

$f \in k[G]$

$\Delta f \in k[G] \otimes k[G] \simeq k[G \times G].$

$(\Delta f)(g, g') = f(gg').$

$\to$ apply to g'

left invariant derivation: $\boxed{\delta(f(g -))(g') = (\delta f)(gg')}$

$\Gamma(G, TG)^G \ni \delta \longmapsto e \circ \delta$; $\mathfrak{g}$ ← derivations at $e \in G$

$\| $

$\mathrm{Der}(k[G])^G$

$\|$

$\left\{ \delta : k[G] \to k[G], \begin{array}{l} \delta \text{ derivation} \\ (id \otimes \delta) \circ \Delta = \Delta \circ \delta \end{array} \right\}.$

$$\mathfrak{g} = \{ d : k[G] \to k \mid df \cdot e(g) + e(f)\, dg = d(fg) \}$$

$$d(fg) = df \cdot \underbrace{g(e)}_{e(g)} + f(e)\, dg$$

$e \in G$

$e : k[G] \longrightarrow k$ counit.

$f \longmapsto f(e)$

$*$ $e \circ \delta \in \mathfrak{g}$. $\quad d : k[G] \to k[G] \xrightarrow{e} k.$

Take $f, g \in k[G]$

$$e \circ \delta\,(fg) = e\big((\delta f) g + f (\delta g) \big)$$
$$= \big(e \circ \delta(f) \big) e(g) + e(f)\, (e \circ \delta)(g)$$

$\Rightarrow e \circ \delta \in \mathfrak{g}$.

$$\mathfrak{g} \longrightarrow \Gamma(G, TG)^G \qquad k[G] \xrightarrow{d} k$$
$$d \longmapsto \underbrace{(id \otimes d) \circ \Delta}_{\delta}. \qquad k[G] \xrightarrow{\Delta} k[G]^{\otimes 2} \xrightarrow{id \otimes d} k[G].$$

$$\delta(fg) = (id \otimes d) \circ \Delta(fg)$$
$$= (id \otimes d)(\Delta f \, \Delta g) \qquad \Delta f = \sum f_i \otimes g_i$$
$$= (id \otimes d)\big(f_{(1)} g_{(1)} \otimes f_{(2)} g_{(2)} \big)$$
$$= f_{(1)} g_{(1)} \otimes \big[e(f_{(2)})\, d g_{(2)} + d f_{(2)} \cdot e(g_{(2)}) \big]$$
$$= \big[\underbrace{f_{(1)} \otimes e(f_{(2)})}_{f} \big] \cdot \underbrace{g_{(1)} \otimes d g_{(2)}}_{\delta(g)} + \big[f_{(1)} \otimes d f_{(2)} \big] \cdot \overbrace{g_{(1)} \otimes e(g_{(2)})}^{g}$$

$\underbrace{(id \otimes e) \circ \Delta(f)}_{id} = f \qquad (id \otimes d) \circ \Delta(f) = \delta(f)$

$= f\, \delta(g) + \delta(f)\, g$. Leibniz rule for δ.

δ is G invariant: $\qquad$ Show $\Delta\circ\delta = (id\otimes\delta)\circ\Delta$

$\overset{\delta}{\|}$ $(id\otimes d)\circ\Delta$

$$\rightsquigarrow \Delta\circ(id\otimes d)\circ\Delta \overset{?}{=} id\otimes\big((id\otimes d)\circ\Delta\big)\circ\Delta.$$

$$\| \quad (id\otimes id\otimes d)\circ\underbrace{(id\otimes\Delta)\circ\Delta}$$

$$(id\otimes id\otimes d)\circ(\Delta\otimes id)\circ\Delta$$

$$\| \quad (\Delta\otimes d)\circ\Delta.$$

$$(\Delta\otimes d)\circ\Delta(c) = c_{(1)}\otimes c_{(2)}\otimes d\,c_{(3)}$$

$$\Delta\circ(id\otimes d)\circ\Delta(c) = \Delta\big(d(c_{(2)})\,c_{(1)}\big)$$

$$= d\,c_{(2)}\cdot (c_{(1)})_{(1)}\otimes (c_{(1)})_{(2)}$$

$$= c_{(1)}\otimes c_{(2)}\;d\,c_{(3)}.$$

OK with left invariance.

$$\mathfrak{g} \longrightarrow \Gamma(G,TG)^G$$

$$d \overset{\varphi}{\longmapsto} (id\otimes d)\circ\Delta$$

$$e\circ\delta \underset{\psi}{\longleftarrow} \delta$$

are inverse to each other.

$$\psi\circ\varphi(d) = e\,(id\otimes d)\circ\Delta.$$

$$= (e\otimes d)\circ\Delta$$

$$= (id\otimes d)\circ\underbrace{(e\otimes id)\circ\Delta}_{id},$$

$$= d.$$

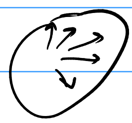
$$\varphi\circ\psi(\delta) = (id\otimes\psi(\delta))\circ\Delta$$

$$= (id\otimes(e\circ\delta))\circ\Delta$$

$$= (id\otimes e)\circ\underbrace{(id\otimes\delta)\circ\Delta}.$$

$$= \underbrace{(id\otimes e)\circ\Delta}_{id}\circ\delta$$

$$= \delta.$$

Lie groups no more intuitive since vector fields .

G Lie group

Lie G = { left invariant vectors fields on G }

$g : G \longrightarrow G$ $\qquad$ $dg(x) : T_x G \longrightarrow T_{gx} G$

$\quad x \longmapsto gx$

X v f on G is left invariant if

$$X(gx) = dg(x)\cdot X(x) \qquad X\cdot f = df(X(-))$$

$$\underset{T_{gx}G}{\overset{}{\cap}} \qquad X\cdot f(x) = df(x)(X(x)).$$

<u>ex</u> 2.7. G - $\qquad TG^{(k)} = G^{(k)} \times \mathfrak{g}^{(k)}$

$\qquad ? \longleftarrow (g, d)$

$\Gamma(G, TG)$ = derivations of $k[G]$.

$k[G] \to k$

$$TG(k) = \mathrm{Hom}_{k\text{-alg}}\left(k[G], k[t]/t^2\right) = \mathrm{Hom}_{k\text{-alg}}(k[G], k) \times \mathfrak{g}$$

$$\underline{\varphi}(f) \mapsto \alpha(f) \oplus t\, d(f) \longleftarrow (\alpha, d)$$

write $k[t]/t^2 = k \oplus tk$ as vector spaces

Show that φ is algebra morphism.

2- $\varphi : G \longrightarrow H$ morphism <u>alg groups</u>

$\varphi^\# : k[H] \longrightarrow k[G]$ $\to$ commutes with Δ with counit.

$d\varphi(e) : \mathfrak{g} \to \mathfrak{h}$ is morphism of Lie algebras.