TD 1 - Thursday, 12 November 2020

B $\qquad (\underset{\text{B-module}}{A}, \overset{\downarrow}{m}, u)$ $\qquad m : A \otimes_B A \to A$

$u : B \to A$

associativity :

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "id \otimes m"] \arrow[d, "m \otimes id"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

unitality :

$$\begin{tikzcd} & B \otimes_B A \arrow[dl, bend right] \arrow[dl, bend left] & & \\ A \otimes_B B \arrow[rr, "id \otimes u"] \arrow[drr] & & A \otimes_B A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\ a \otimes b \arrow[r, mapsto] & ba & A \end{tikzcd}$$

$\Rightarrow$ $m(a \otimes u(b)) \overset{\shortparallel}{=} ba$ and $m(u(b) \otimes a) \overset{\shortparallel}{=} ba$

so $u(b)$ commutes with any elt of A.

(C, Δ, ε) $\qquad C$ B-module

$\Delta : C \to C \otimes_B C$

$\varepsilon : C \to B$.

coassociativity

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & C \otimes_B C \arrow[d, "id \otimes \Delta"] \\ C \otimes_B C \arrow[r, "\Delta \otimes id"'] & C \otimes_B C \otimes_B C \end{tikzcd}$$

counitality

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\sim"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & A \otimes_B A \arrow[d, "\varepsilon \otimes id"] \\ & B \otimes_B A \arrow[loop below] \end{tikzcd}$$

$id \otimes \varepsilon$.

B - bialgebra A. $(A, m, u, \Delta, \varepsilon)$

(A, m, u) algebra

(A, Δ, ε) coalgebra

$$m : \overparen{A \otimes_B A} \longrightarrow A$$

$$\Delta : A \longrightarrow \underbrace{A \otimes_B A}_{\text{algebra structure}}$$ algebra morphism.

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'.$$

$$\varepsilon : A \longrightarrow B$$ algebra morphism

[coalgebra structure on $A \otimes_B A$

$$\overset{A'}{\overset{\shortparallel}{A \otimes_B A}} \xrightarrow[\tau_{23} \circ \Delta \otimes \Delta]{} A' \otimes_B A'$$

$$\tau_{23} : A^{\otimes 4} \longrightarrow A^{\otimes 4}$$

$$a \otimes \underbrace{b \otimes c}_{} \otimes d \mapsto a \otimes \underbrace{c \otimes b}_{} \otimes d$$]

algebra
coalgebra] → bialgebra

Hopf algebra : $(\underbrace{A, m, u, \Delta, \varepsilon}_{\text{bialgebra}}, S)$

S "antipode"

$S : A \longrightarrow A$ of B-modules.

$$m \circ (S \otimes id) \circ \Delta = u \circ \varepsilon$$

$$\begin{tikzcd} & A \arrow[dl, "\varepsilon"'] \arrow[r, "\Delta"] & A \otimes A \arrow[dd, "S \otimes id"] \\ B \arrow[dr, "u"'] & \circlearrowright & \\ & A & A \otimes A \arrow[l, "m"] \end{tikzcd}$$

$\boxed{1.1}$ $$\Delta(c) = \sum_{i\in I} c_i \otimes c_i'$$
$$= c_{(1)} \otimes c_{(2)}$$

1- $(id \otimes \Delta)\circ\Delta = (\Delta\otimes id)\circ\Delta. =: \Delta^{(2)}$

apply to c $\downarrow$ $\qquad$ $\Delta^{(n)}$ in the same way

$$(id\otimes\Delta)(c_{(1)}\otimes c_{(2)}) = (\Delta\otimes id)(c_{(1)}\otimes c_{(2)})$$
$$\qquad\qquad || \qquad\qquad\qquad\qquad\qquad ||$$

coassociativity $\rightarrow$ $$\boxed{c_{(1)}\otimes (c_{(2)})_{(1)}\otimes (c_{(2)})_{(2)} = (c_{(1)})_{(1)}\otimes(c_{(1)})_{(2)}\otimes c_{(2)}}$$

Sweedler : $\Delta^{(2)}(c) = c_{(1)}\otimes c_{(2)}\otimes c_{(3)}$.
$$\Delta^{(n)}(c) = \bigotimes_{i=1}^{n} c_{(i)} .$$

counitality : $\overset{\leftrightarrow}{(\varepsilon\otimes id)}\otimes\Delta = id$

apply to c $\boxed{\varepsilon(c_{(1)})c_{(2)} = c}$

$$\varepsilon(c_{(1)})c_{(2)} = c_{(1)}\,\varepsilon(\underbrace{c_{(2)}}_{\Uparrow\, B}) = c$$

2-
$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & C' \arrow[d, "\Delta'"] \\
C\otimes C \arrow[r, "f\otimes f"'] & C'\otimes C'
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & \circlearrowright & C' \arrow[dl, "\varepsilon'"] \\
& B &
\end{tikzcd}$$

$$\varepsilon'\circ f = \varepsilon$$

$c\in C$ $$\boxed{f(c)_{(1)}\otimes f(c)_{(2)} = f(c_{(1)})\otimes f(c_{(2)}).}$$

[H, H' Hopf $f : H \longrightarrow H'$ bialgebras.
→ automatically preserves the antipode.

3- A

m and Δ

$$\Delta \circ m_A = m_{A \otimes_B A} \circ \Delta \otimes \Delta$$

apply to $a \in A$
$b \in B$

$$\underset{\substack{\| \\ \boxed{(ab)_{(1)} \otimes (ab)_{(2)}}}}{\Delta(ab)} = \underset{\substack{\| \\ \boxed{a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}}}}{m_{A \otimes_B A}} \Big(\overset{\substack{a_{(1)} \otimes a_{(2)} \\ \|}}{\Delta(a)} \otimes \overset{\substack{b_{(1)} \otimes b_{(2)} \\ \|}}{\Delta(b)} \Big)$$

$$\begin{tikzcd}
A & B \arrow[l, "u"'] \\
A \otimes A \arrow[from=1-1, "\Delta_A"'] & B \arrow[from=1-2, "\Delta_B = id_B"] \arrow[l] \\
u(b) \otimes u(b) & b \arrow[l, mapsto]
\end{tikzcd}$$

$B \otimes_B B = B$

bi

B is a B- coalgebra

with $\Delta_B : B \longrightarrow B \otimes_B B = B$
$\overset{\|}{id_B}$.

$\Delta_A \circ u = (u \otimes u) \circ \Delta_B$] $\varepsilon : B \xrightarrow{id} B$

$u(b)_{(1)} \otimes u(b)_{(2)} = u(b) \otimes u(b)$.]

$b = 1$. $\Delta(1) = 1 \otimes 1$.

$$\begin{tikzcd}
B \arrow[r, "u"] \arrow[dr, equal] & A \arrow[d, "\varepsilon"] \\
 & B
\end{tikzcd}$$

$$\boxed{\varepsilon \circ u = id_B.}$$

ex 1.2 : $\Delta : A \to A \otimes_B A$ $\qquad \tau : A \otimes_B A \longrightarrow A \otimes_B A$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a \otimes b \longmapsto b \otimes_B a$

A cocommutative if $\Delta = \tau \circ \Delta$

m commutative : $m \circ \tau = m$.

1- $\mathrm{Hom}_B(\overset{\text{coalgebra}}{C}, \underset{\text{algebra}}{A})$

$$f * g = m_A \circ (f \otimes g) \circ \Delta_c$$

$$f * (g * h) = m_A \circ (f \otimes (g * h)) \circ \Delta_c$$

$$= m_A \circ \big(\underbrace{f \otimes (\underbrace{m_A} \circ (g \otimes h) \circ \underbrace{\Delta_c})}_{\shortparallel} \big) \circ \Delta_c$$

$$= m_A \circ \big(\overbrace{(id \otimes m_A) \circ (f \otimes g \otimes h) \circ (id \otimes \Delta_c)} \big) \circ \Delta_c$$

$$= (m_A \circ (id \otimes m_A)) \circ (f \otimes g \otimes h) \circ ((id \otimes \Delta_c) \circ \Delta_c)$$

$(f * g) * h = \cdots$ $\qquad\qquad$ OK for associativity of $*$.

2- $\underset{\substack{\Psi \\ u \circ \varepsilon}}{\mathrm{Hom}_B(C, A)}$ $\qquad\qquad (C, \Delta, \varepsilon)$

$\qquad\qquad\qquad\qquad\qquad\qquad (A, m, u)$

$$C \xrightarrow{\varepsilon} B \xrightarrow[u]{} A \qquad u \circ \varepsilon$$

$\mathbb{1} = u \circ \varepsilon$

$f \in \mathrm{Hom}_B(C, A)$

$$f * \mathbb{1} = m_A \circ \underbrace{\left(f \otimes (u \circ \varepsilon) \right)} \circ \Delta_C \qquad \left(\overset{?}{=} f \right)$$

$$= \underbrace{m_A \circ (id \otimes u)}_{id_A} \circ \overbrace{\underbrace{\left(f \otimes id_B\right)}_{\overset{\shortparallel}{f}} \circ \underbrace{(id \otimes \varepsilon) \circ \Delta_C}_{id_C}}^{\shortparallel}$$

$$= f.$$

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, phantom, "\cong"] & C \arrow[d, phantom, "\cong"] \\ C \otimes_B B \arrow[r] & C \otimes_B B \end{tikzcd}$$

$f \otimes id_B : C \otimes_B B \longrightarrow C \otimes_B B$

also $\mathbb{1} * f = f$ → $\mathbb{1}$ is the unit.

3. A halgebra $\mathrm{Hom}_B(A,A) \ni id_A$ has an algebra structure by convolution.

S antipode

$$S * id_A = m_A \circ (S \otimes id_A) \circ \Delta_A$$
$$= u \circ \varepsilon$$
$$= \mathbb{1}$$

$$id_A * S = \mathbb{1}$$

4- A

$$\begin{tikzcd} A \otimes_B A \arrow[rr, "\varepsilon_{A \otimes_B A}"] \arrow[dr, "m_A"'] & & B \\ & A \arrow[ur, "\varepsilon_A"'] & \end{tikzcd}$$

$A' = A \otimes_B A$

$$\varepsilon_{A \otimes_B A} = \varepsilon_A \circ m_A.$$

$$\Delta_{A \otimes_B A} : A' \xrightarrow[\tau_{23} \circ \Delta \otimes \Delta]{} A' \otimes_B A'$$

→ Check $(A', \Delta_{A'}, \varepsilon_{A'})$ is a coalgebra.

5- $S : A \longrightarrow A$ is antihomomorphism

$\forall a, b\ ,\ S(ab) = S(b)\, S(a).$

$\mathcal{E} = \mathrm{Hom}_B(\underset{\text{coalgebra}}{\underbrace{A \otimes_B A}}, \underset{\text{algebra}}{\underbrace{A}}) \rightarrow$ convolution product. $*$.

$$\mathcal{E} \ni \underset{a\otimes b \mapsto ab}{m_A}\ ,\quad \underset{a\otimes b \mapsto S(ab)}{\overset{\rho}{\overset{\shortparallel}{S\circ m_A}}}\ ,\quad \underset{a\otimes b \mapsto S(b)S(a).}{\overset{\mu}{\overset{\shortparallel}{m_A \circ \tau \circ (S\otimes S)}}}$$

S antihomomorphism $\iff$ $\rho = \mu$.

We show : ρ is an inverse of m in $\mathcal{E}$

μ ——————— $\mathcal{E}$

$\Rightarrow \rho = \mu.$

Show

$\rho * m = \underset{u\circ\varepsilon}{\overset{\shortparallel}{\mathbb{1}}}$

$m * \rho$

$m * \mu$

$\mu * m.$

$$\rho * m = m_A \circ (\underset{S\circ m_A}{\overset{}{\underbrace{\rho}_{\shortparallel}}} \otimes m_A) \circ \Delta_{A\otimes_B A}$$

$$= m_A \circ \underbrace{\left((S\circ m_A) \otimes m_A \right)} \circ \Delta_{A'}$$

$$= m_A \circ \left((S\otimes id_A) \circ \underbrace{m_A \otimes m_A) \circ \Delta_{A'}} \right.$$

$$\begin{tikzcd}
A^{\otimes 2} \arrow[r, "\Delta_{A'}"] \arrow[d, "m_A"'] & A^{\otimes 4} \arrow[d, "m_A \otimes m_A"] \\
A \arrow[r, "\Delta_A"'] \arrow[ur, phantom, "\circlearrowright"] & A^{\otimes 2}
\end{tikzcd}$$

$$= \underbrace{m_A \circ (S\otimes id_A) \circ \Delta_A} \circ m_A$$

$$= \mu \circ \underbrace{\varepsilon \circ m_A}_{\varepsilon_{A'}}$$

$$= u_A \circ \varepsilon_{A'} = \mathbb{1} \in \mathcal{E}.$$

$$(\mu * m_A)(a\otimes b) = m_A \circ (\mu \otimes m_A) \circ \underbrace{\Delta_{A'} \, (a\otimes b)}$$

$$\parallel$$

$$\underbrace{a_{(1)} \otimes b_{(1)}}_{\mu} \otimes \underbrace{a_{(2)} \otimes b_{(2)}}_{m_A}$$

$$= m_A\big(S(b_{(1)})\, S(a_{(1)}) \otimes a_{(2)}\, b_{(2)} \big)$$

$$= S(b_{(1)}) \underbrace{S(a_{(1)})\, a_{(2)}}\, b_{(2)}$$

$$\parallel$$

$$m_A \circ (S\otimes id) \circ \Delta_A(a)$$

$$\parallel$$

$$u \circ \varepsilon(a)$$

$$= S(b_{(1)})\, (u\circ\varepsilon(a))\, b_{(2)} \qquad (u : B \longrightarrow A)$$

$$= u\circ\varepsilon(a)\; u\circ\varepsilon(b)$$

$$= u\circ\varepsilon(ab) = u\circ \underbrace{\varepsilon \circ m_A}_{\varepsilon_{A'}} (a\otimes b)$$

$$= \mathbb{1}(a\otimes b)$$

$\Rightarrow$ μ is left inverse to m_A in $\mathcal{C}$.

$\rightarrow$ $\mu = m_A^{-1}$ in $\mathcal{C}$
$\rho = m_A^{-1}$ in $\mathcal{C}$ $\quad \Rightarrow \quad \mu = \rho$

6- $\quad 1) \Rightarrow 2)$

Assume $S^2 = id$

$\forall h \in H$, $\quad S(h_{(1)})\, h_{(2)} = u\circ\varepsilon(h)$ ⊛

$S \circ u = u.$

apply S : $\quad S(h_{(2)})\, \underbrace{S^2(h_{(1)})}_{id} = \underbrace{S\circ u}_{u} \circ\, \varepsilon(h).$

[for $h = 1$ $\quad \Delta(h) = 1 \otimes 1 \quad \Rightarrow \quad \overset{S(1_A)}{S(1)} \cdot 1_A = u\circ\varepsilon(1)$
$= 1_A$]

$$S(h_{(2)})\, h_{(1)} = u \circ \varepsilon(h).$$

$2) \Rightarrow 1)$. S is inverse to id_A in $Hom_B(A,A)$. (with convolution product)

To show $S^2 = id$, we prove that S^2 is inverse to S in $Hom_A(A,A)$

$$\begin{aligned}(S * S^2)(h) &= m_A \circ (S \otimes S^2) \circ \Delta_A(h)\\ &= S(h_{(1)})\, S^2(h_{(2)})\\ &= S(\underbrace{S(h_{(2)})\, h_{(1)}}_{\overset{=}{\text{by } 2.}\ u \circ \varepsilon(h)}).\\ &= u \circ \varepsilon(h) = \mathbb{1}(h).\end{aligned}$$

Similarly, $S^2 * S = \mathbb{1} \Rightarrow S^2 = id_A$.

$(1) \Leftrightarrow (3)$ is analogous

7- Assume H is cocommutative:

$\forall h \in H \quad h_{(1)} \otimes h_{(2)} = h_{(2)} \otimes h_{(1)}$.

$$\rightarrow \quad \underset{m_A \circ (S \otimes id) \circ \Delta_A(h)}{\overset{\shortparallel}{S(h_{(1)})\, h_{(2)}}} = u \circ \varepsilon(h) \underset{\uparrow}{=} S(h_{(2)})\, h_{(1)}.$$

$2. \Rightarrow \underline{S^2 = id}$.

Assume H is commutative

$$\forall h \in H, \quad \underset{h_{(2)}\, S(h_{(1)})}{\overset{\shortparallel}{S(h_{(1)}) \cdot h_{(2)}}} = u \circ \varepsilon(h) \ \ \underset{\uparrow}{=} $$

3. of Q.6

$$\Rightarrow \boxed{S^2 = id}$$

$\mathcal{U}(sl_2)$ = enveloping algebra of sl_2

sl_2 = Lie alg of traceless matrices in
$gl_2 = 2 \times 2$ matrices (say over $\mathbb{Q}$)

$sl_2 = e, f, h$

$\mathcal{U}(sl_2)$

$S^2(E) = q^2 E \neq E$ so $S^2 \neq id_{\underline{\mathcal{U}_q(sl_2)}}$.
in $\mathcal{U}_q(sl_2)$

<u>ex 1.3</u>: X $\overset{}{=}$ algebraic variety, $k = \bar{k}$
(integral, connected scheme of finite type over k)

Then $\forall \underset{\text{open}}{U \subset X}$, $f \in \mathcal{O}_X(U)$ is determined by

its values at k-points. (Nullstellensatz)

$\boxed{X(k)}$ $f : X(k) \longrightarrow k$, ...

G algebraic group. $a \in G(k)$ acts on G by translation
$m : G \times G \longrightarrow G$
+ unit

$G /_{\mathrm{Spec}\, B}$ scheme B algebra $G = \mathrm{Spec}(A)$

$\rightarrow$ G determined by the functor of points

$$\begin{tikzcd}
(\mathrm{AffSch}/_B)^{op} \arrow[r] \arrow[ddd, bend right, "G(-)"'] & \mathrm{Sets} \\
\mathrm{Spec}\, B' \arrow[r, mapsto] & \mathrm{Hom}(\mathrm{Spec}\, B', G) = G(B') \\
 & \overset{\shortparallel}{\mathrm{Hom}_{B\text{-alg}}(A, B')} \\
 & \mathrm{Grps} \arrow[uuu, bend right, "\mathrm{For}"']
\end{tikzcd}$$

G group scheme if

$g \in \mathrm{Hom}_{B\text{-algebras}}(A,B) \rightsquigarrow g_{B'} \in \mathrm{Hom}(A,B') \in G(B')$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[rr, bend right, "g_{B'}"'] & B \arrow[r] & B' \end{tikzcd}$$

$\|$

Then, any $g \in G(B)$ acts on G by translation

B' a B algebra

$$\begin{tikzcd} G(B') \arrow[r, "g_{B'}\cdot"] \arrow[d, equal] & G(B') \arrow[d, equal] \\ \mathrm{Hom}_{B\text{-alg}}(A,B') & \mathrm{Hom}_{B\text{-alg}}(A,B') \end{tikzcd}$$

↑ group ← group

$(B=k)$

G/k $G(k) \circlearrowright G$ by left/right translation

G algebraic variety:

G ← is not a group ——— The underlying set of the underlying top space.

[$G(k)$ is a group. (abstract) but not topological)

If $k=\bar{k}$, X alg. variety, $\underbrace{|X|}_{\text{closed points}} = X(k)$

- * SGA3
- * Waterhouse group schemes
- * Milne, algebraic groups over k not alg closed.

$\mathbb{Q}(q) = \mathrm{Frac}(\underbrace{\mathbb{Q}[q]}_{\text{polynomials in } q}) =$ rational fractions in q.