

TD 1 - Thursday, 12 November 2020

B (A, m, u) $m: A \otimes_B A \rightarrow A$
 B -module $u: B \rightarrow A$

associativity:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{id \otimes m} & A \otimes A \\ m \otimes id \downarrow & \curvearrowright & \downarrow m \\ A \otimes A \otimes A & \xrightarrow{m} & A \end{array}$$

unitality:

$$\begin{array}{ccc} A \otimes_B B & \xrightarrow{id \otimes u} & A \otimes_B A \\ a \otimes b \searrow & \Downarrow & \downarrow m \\ & & A \end{array} \Rightarrow \begin{array}{l} m(a \otimes u(b)) \\ \text{"} \\ ba \\ \text{and} \\ m(u(b) \otimes a) \\ \text{"} \\ ba \end{array}$$

so $u(b)$ commutes with any elt of A .

(C, Δ, ϵ) C B -module
 $\Delta: C \rightarrow C \otimes_B C$
 $\epsilon: C \rightarrow B$

coassociativity

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes_B C \\ \Delta \downarrow & \curvearrowright & \downarrow id \otimes \Delta \\ C \otimes_B C & \xrightarrow{\Delta \otimes id} & C \otimes_B C \otimes_B C \end{array}$$

counitality

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes_B A \\ \searrow \sim & \Downarrow & \downarrow \epsilon \otimes id \\ & & B \otimes_B A \end{array} \quad id \otimes \epsilon$$

B-algebra A . $(A, m, \mu, \Delta, \epsilon)$
 (A, m, μ) algebra
 (A, Δ, ϵ) coalgebra

$$m : \underbrace{A \otimes_B A}_B \rightarrow A$$

$$\Delta : A \rightarrow \underbrace{A \otimes_B A}_B \text{ algebra morphism.}$$

algebra structure

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

$$\epsilon : A \rightarrow B \text{ algebra morphism}$$

[coalgebra structure on $A \otimes_B A$

$$\begin{array}{ccc} A' & & \\ \parallel & & \\ A \otimes_B A & \longrightarrow & A' \otimes_B A' \\ \tau_{23} \circ \Delta \otimes \Delta & & \end{array}$$

$$\tau_{23} : A^{\otimes 4} \rightarrow A^{\otimes 4}$$

$$a \otimes b \otimes c \otimes d \mapsto a \otimes c \otimes b \otimes d$$

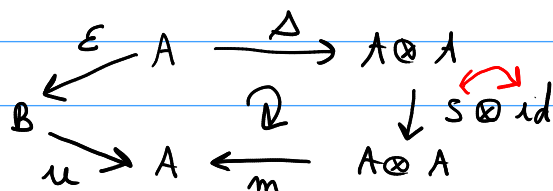
algebra }
 coalgebra } → bialgebra

Hopf algebra : $(A, m, \mu, \Delta, \epsilon, S)$
 bialgebra

S "antipode"

$S : A \rightarrow A$ of B -modules.

$$m \circ (S \otimes id) \circ \Delta = \mu \circ \epsilon$$



1.1

$$\Delta(c) = \sum_{i \in I} c_i \otimes c_i'$$

$$= c_{(1)} \otimes c_{(2)}$$

1- $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta. =: \Delta^{(2)}$
apply to c $\Delta^{(n)}$ in the same way

$$(id \otimes \Delta)(c_{(1)} \otimes c_{(2)}) = (\Delta \otimes id)(c_{(1)} \otimes c_{(2)})$$

// //

coassociativity
 $\rightarrow c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)} = (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)}$

Sweedler: $\Delta^{(2)}(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$
 $\Delta^{(n)}(c) = \bigotimes_{i=1}^n c_{(i)}$

co-unitarity: $(\epsilon \otimes id) \circ \Delta = id$

apply to c $\epsilon(c_{(1)}) c_{(2)} = c$

$$\epsilon(c_{(1)}) c_{(2)} = c_{(1)} \underbrace{\epsilon(c_{(2)})}_{\hat{B}} = c$$

2-

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ \Delta \downarrow & \curvearrowright & \downarrow \Delta' \\ c \otimes c & \xrightarrow{f \otimes f} & c' \otimes c' \end{array}$$

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ \epsilon \searrow & \downarrow B & \swarrow \epsilon' \\ & B & \end{array}$$

$\epsilon' \circ f = \epsilon$

$c \in C$

$$f(c)_{(1)} \otimes f(c)_{(2)} = f(c_{(1)}) \otimes f(c_{(2)})$$

$\begin{matrix} H \\ H' \end{matrix}$ Hopf $f: H \rightarrow H'$ bialgebras.
 \rightarrow automatically preserves the antipode.

3- A
 m and Δ

$$\Delta \circ m_A = m_{A \otimes A} \circ \Delta \otimes \Delta$$

apply to $a \in A$
 $b \in B$

$$\Delta(ab) = m_{A \otimes A} \left(\overset{a_{(1)} \otimes a_{(2)}}{\Delta(a)} \otimes \overset{b_{(1)} \otimes b_{(2)}}{\Delta(b)} \right)$$

$$\boxed{(ab)_{(1)} \otimes (ab)_{(2)} = a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}}$$

$$\begin{array}{ccc} A & \xleftarrow{u} & B \\ \Delta_A \downarrow & & \downarrow \Delta_B = id_B \\ A \otimes A & \xleftarrow{} & B \\ u(b) \otimes u(b) & \xleftarrow{} & b \end{array}$$

$B \otimes_B B = B$
 B is a B -coalgebra
 with $\Delta_B: B \rightarrow B \otimes_B B = B$
 id_B .

$$\Delta_A \circ u = (u \otimes u) \circ \Delta_B \quad \varepsilon: B \xrightarrow{id} B$$

$$u(b)_{(1)} \otimes u(b)_{(2)} = u(b) \otimes u(b).$$

$b=1. \quad \Delta(1) = 1 \otimes 1.$

$$\begin{array}{ccc} B & \xrightarrow{u} & A \\ & \searrow \varepsilon & \downarrow \varepsilon \\ & & B \end{array}$$

$$\boxed{\varepsilon \circ u = id_B.}$$

ex 1.2: $\Delta: A \rightarrow A \otimes_B A$ $\tau: A \otimes_B A \rightarrow A \otimes_B A$
 $a \otimes b \mapsto b \otimes a$
 A cocommutative if $\Delta = \tau \circ \Delta$
 m commutative: $m \circ \tau = m$

1- $\text{Hom}_B(C, A)$
 coalgebra \leftarrow algebra

$$f * g = m_A \circ (f \otimes g) \circ \Delta_C$$

$$\begin{aligned} f * (g * h) &= m_A \circ (f \otimes (g * h)) \circ \Delta_C \\ &= m_A \circ (f \otimes (m_A \circ (g \otimes h) \circ \Delta_C)) \circ \Delta_C \\ &= m_A \circ ((\text{id} \otimes m_A) \circ (f \otimes g \otimes h) \circ (\text{id} \otimes \Delta_C)) \circ \Delta_C \\ &= (m_A \circ (\text{id} \otimes m_A)) \circ (f \otimes g \otimes h) \circ ((\text{id} \otimes \Delta_C) \circ \Delta_C) \end{aligned}$$

$$(f * g) * h = \dots$$

OK for associativity of $*$.

2- $\text{Hom}_B(C, A)$ (C, Δ, ε)
 $\downarrow \cup$ (A, m, μ)
 $\mu \circ \varepsilon$
 $C \xrightarrow{\varepsilon} B \xrightarrow{\mu} A \quad \mu \circ \varepsilon$

$$\mathbb{1} = \mu \circ \varepsilon$$

$$f \in \text{Hom}_B(C, A)$$

$$\begin{aligned}
 f * 1 &= m_A \circ (f \otimes (\mu \circ \varepsilon)) \circ \Delta_C \quad \left(\stackrel{?}{=} f \right) \\
 &= m_A \circ \underbrace{(id \otimes \mu)}_{id_A} \circ \underbrace{(f \otimes id)}_f \circ \underbrace{(id \otimes \varepsilon)}_{id_C} \circ \Delta_C \\
 &= f.
 \end{aligned}$$

$C \xrightarrow{f} C$
 $\parallel S \quad \parallel S$
 $f \otimes id_B : C \otimes_B B \rightarrow C \otimes_B B$

also $1 * f = f \rightarrow 1$ is the unit.

3. A algebra $\text{Hom}_B(A, A)$ has an algebra structure by convolution.

\cup
 id_A

S antipode

$$\begin{aligned}
 S * id_A &= m_A \circ (S \otimes id_A) \circ \Delta_A \\
 &= \mu \circ \varepsilon \\
 &= 1
 \end{aligned}$$

$$id_A * S = 1$$

4- A

$$A' = A \otimes_B A$$

$$\begin{array}{ccc}
 A \otimes_B A & \xrightarrow{\varepsilon_{A \otimes_B A}} & B \\
 & \searrow m_A & \nearrow \varepsilon_A \\
 & A &
 \end{array}$$

$$\varepsilon_{A \otimes_B A} = \varepsilon_A \circ m_A.$$

$$\Delta_{A \otimes_B A} : A' \longrightarrow A' \otimes_B A' \\
 \tau_{23} \circ \Delta \otimes \Delta$$

\rightarrow Check $(A', \Delta_{A'}, \varepsilon_{A'})$ is a coalgebra.

5- $S: A \rightarrow A$ is antihomomorphism
 $\forall a, b, S(ab) = S(b)S(a).$

$\mathcal{E} = \text{Hom}_B(A \otimes_B A, A) \rightarrow$ convolution product. \times .

$\underbrace{\quad}_{\text{coalgebra}} \quad \underbrace{\quad}_{\text{algebra}}$

$$\mathcal{E} \ni m_A, \quad \overset{\rho}{S \circ m_A}, \quad \overset{\mu}{m_A \circ S \circ (S \otimes S)}$$

$$a \otimes b \mapsto ab \quad a \otimes b \mapsto S(ab) \quad a \otimes b \mapsto S(b)S(a).$$

S antihomomorphism $\iff \rho = \mu.$

We show: ρ is an inverse of m in \mathcal{E}

$\mu \quad \text{-----} \quad \mathcal{E}$

$$\implies \rho = \mu.$$

Show

$$\rho * m = \mathbb{1} \quad \mu \in \mathcal{E}$$

$$m * \rho$$

$$m * \mu$$

$$\mu * m.$$

$$\rho * m = m_A \circ \left(\underbrace{\rho \otimes m_A}_{S \circ m_A} \right) \circ \Delta_{A \otimes_B A}$$

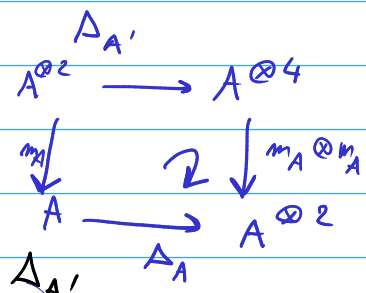
$$= m_A \circ \left((S \circ m_A) \otimes m_A \right) \circ \Delta_{A'}$$

$$= m_A \circ \left((S \otimes \text{id}_A) \circ m_{A \otimes_B A} \right) \circ \Delta_{A'}$$

$$= m_A \circ (S \otimes \text{id}_A) \circ \underbrace{\Delta_{A'} \circ m_A}_{\mathbb{1}}$$

$$= \underbrace{\mu \circ \mathcal{E} \circ m_A}_{\mathcal{E}_{A'}}$$

$$= \mu_A \circ \mathcal{E}_{A'} = \mathbb{1} \in \mathcal{E}.$$



$$\begin{aligned}
(\mu * m_A)(a \otimes b) &= m_A \circ (\mu \otimes m_A) \circ \Delta_{A'}(a \otimes b) \\
&= m_A \left(\underbrace{S(b_{(1)}) S(a_{(1)})}_{\mu} \otimes \underbrace{a_{(2)} b_{(2)}}_{m_A} \right) \\
&= S(b_{(1)}) \underbrace{S(a_{(1)}) a_{(2)}}_{m_A \circ (S \otimes id) \circ \Delta_{A'}(a)} b_{(2)} \\
&= S(b_{(1)}) (m_A \circ \epsilon(a)) b_{(2)} \quad (\mu: B \rightarrow A) \\
&= m_A \circ \epsilon(a) m_A \circ \epsilon(b) \\
&= m_A \circ \epsilon(ab) = m_A \circ \underbrace{\epsilon \circ m_A}_{\epsilon_{A'}}(a \otimes b) \\
&= \mathbb{1}(a \otimes b)
\end{aligned}$$

$\Rightarrow \mu$ is left inverse to m_A in \mathcal{E} .

$$\begin{aligned}
\rightarrow \mu &= m_A^{-1} \text{ in } \mathcal{E} \\
\mu &= m_A^{-1} \text{ in } \mathcal{E} \quad \Rightarrow \mu = \rho
\end{aligned}$$

6- 1) \Rightarrow 2)

Assume $S^2 = id$.

$$\forall h \in H, \quad S(h_{(1)}) h_{(2)} = m_A \circ \epsilon(h)$$

$$\text{apply } S : \quad S(h_{(2)}) \underbrace{S^2(h_{(1)})}_{id} = \underbrace{S \circ m_A}_{\mu} \circ \epsilon(h).$$

$$\left[\text{for } h = 1 \quad \Delta(1) = 1 \otimes 1 \Rightarrow S(1) \cdot 1_A = m_A \circ \epsilon(1) \right. \\
\left. \begin{array}{l} \parallel \\ S(1_A) = 1_A \end{array} \right]$$

$$S(h_{(2)}) h_{(1)} = \mu \circ \varepsilon(h).$$

2) \Rightarrow 1). S is inverse to id_A in $\text{Hom}_B(A, A)$. (with counit product)

To show $S^2 = \text{id}$, we prove that S^2 is inverse to S in $\text{Hom}_A(A, A)$

$$\begin{aligned} (S * S^2)(h) &= m_A \circ (S \otimes S^2) \circ \Delta_A(h) \\ &= S(h_{(1)}) S^2(h_{(2)}) \\ &= S\left(\underbrace{S(h_{(2)}) h_{(1)}}_{\substack{= \mu \circ \varepsilon(h) \\ \text{by 2.}}}\right) \\ &= \mu \circ \varepsilon(h) = \mathbb{1}(h). \end{aligned}$$

Similarly, $S^2 * S = \mathbb{1} \Rightarrow S^2 = \text{id}_A$.

(1) \Rightarrow (3) is analogous

7- Assume H is cocommutative:

$$\forall h \in H, \quad h_{(1)} \otimes h_{(2)} = h_{(2)} \otimes h_{(1)}. \quad \Leftarrow$$

$$\begin{aligned} \rightarrow S(h_{(1)}) h_{(2)} &= \mu \circ \varepsilon(h) \stackrel{1.}{=} S(h_{(2)}) h_{(1)}. \\ \parallel & \qquad \qquad \qquad \uparrow \\ m_A \circ (S \otimes \text{id}) \circ \Delta_A(h) & \qquad \qquad \qquad 2. \Rightarrow \underline{S^2 = \text{id}}. \end{aligned}$$

Assume H is commutative

$$\forall h \in H, \quad S(h_{(1)}) \cdot h_{(2)} = \mu \circ \varepsilon(h)$$

$$\parallel \qquad \qquad \qquad \uparrow \text{3. of Q.6}$$

$$\Rightarrow \boxed{S^2 = \text{id}}$$

$U(\mathfrak{sl}_2) =$ enveloping algebra of \mathfrak{sl}_2

$\mathfrak{sl}_2 =$ Lie alg of traceless matrices in $\mathfrak{gl}_2 = 2 \times 2$ matrices (say over \mathbb{C})

$\mathfrak{sl}_2 = e, f, h$

$U(\mathfrak{sl}_2)$

$S^2(E)$ in $U_q(\mathfrak{sl}_2) = q^2 E \neq E$ so $S^2 \neq \text{id}$ $U_q(\mathfrak{sl}_2)$.

ex 1.3: X $k = \bar{k}$
 \parallel
 algebraic variety $\left(\begin{array}{l} \text{integral, connected scheme of finite type} \\ \text{over } k \end{array} \right)$

Then $\forall U \subset X$ open, $f \in G_X(U)$ is determined by

its values at k -points. (Nullstellensatz)

$X(k)$ $f : X(k) \rightarrow k \dots$

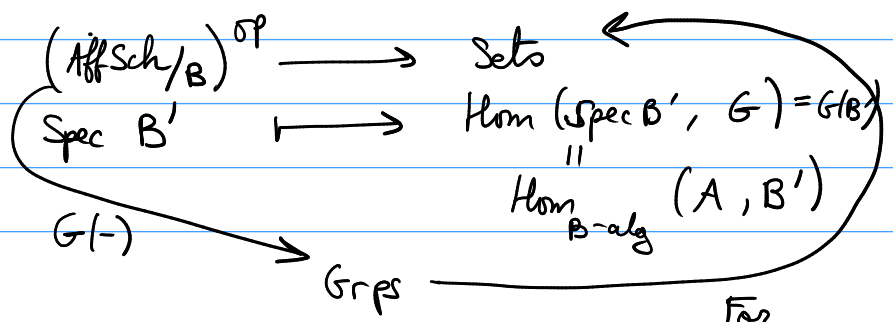
G algebraic group.

$m : G \times G \rightarrow G$
 $+ \text{unit}$

$a \in G(k)$ acts on G by translation

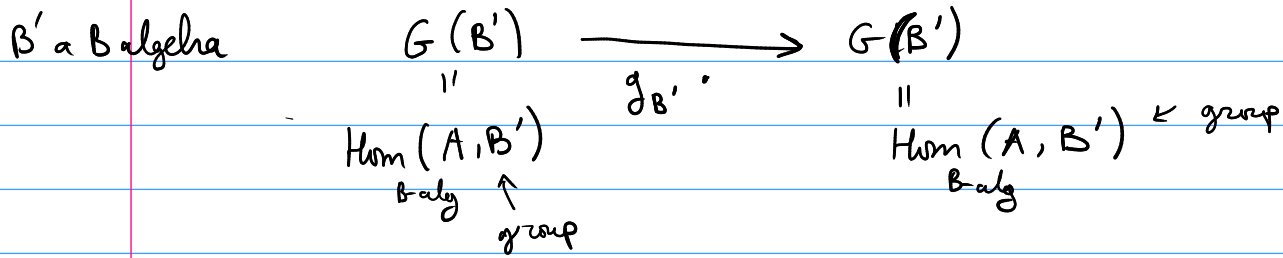
$G / \text{Spec } B$ scheme B algebra $G = \text{Spec}(A)$

$\rightarrow G$ determined by the functor of points



G group scheme of

$g \in \text{Hom}(A, B) \rightsquigarrow g_{B'} \in \text{Hom}(A, B') \in G(B')$
 \parallel B -algebra \parallel $A \xrightarrow{\theta} B \xrightarrow{\theta'} B'$
 Then, any $g \in G(B)$ acts on G by translation



$(B=k)$

$G/k \quad G(k) \curvearrowright G$
 by left/right translation

G algebraic variety:

G is not a group the underlying set of the underlying top space.
 $[G(k) \text{ is a group. (abstract) but not topological}]$

If $k = \bar{k}$, X alg. variety, $|X| = X(k)$
closed points

* SGA3

* Waterhouse group schemes

* Milne, algebraic groups over k not alg closed.

$\mathcal{Q}(q) = \text{Frac}(\mathcal{Q}[q]) = \text{rational fractions in } q$
polynomials in q