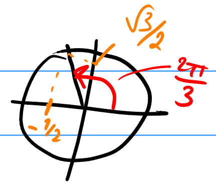
TD 16 - Lundi 7 décembre 2020



exercice 11 : Résoudre $z^2+z+1=0$

Trouver les valeurs $\cos\left(\frac{2\pi}{3}\right)$ et $\sin\left(\frac{2\pi}{3}\right)$

$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$; $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt3}{2}$.

$\sqrt{-5} = i\sqrt5$
$i^2 = -1$.

$\Delta = +1-4 = -3$

$$z_1 = \frac{-1-\sqrt\Delta}{2} = \frac{-1-i\sqrt3}{2}$$

$$z_2 = \frac{-1+\sqrt\Delta}{2} = \frac{-1+i\sqrt3}{2}$$

- pas futé : $z_1^3 = \left(\frac{-1-i\sqrt3}{2}\right)^3 = -\frac{1}{8}(1+i\sqrt3)^3 = -\frac18\overbrace{\left(1+\cancel{3i\sqrt3}+\underbrace{3\cdot i^2\cdot 3}_{-9}+\cancel{i^3(\sqrt3)^3}\right)}^{-8} = 1$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

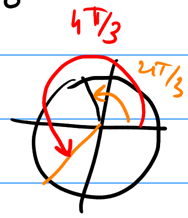
$(a+b)^2 = a^2+2ab+b^2$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
⋮

[triangle de Pascal]

$z^3-1 = (z-1)(z^2+z+1)$.

Donc en particulier, $z_1^3-1 = 0 = z_2^3-1$. donc z_1, z_2 sont des racines cubiques de 1.



3- Les racines de 1 sont $1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}}$.

en part, $\mathrm{Im}(e^{i2\pi/3}) > 0$
$\mathrm{Im}(e^{i4\pi/3}) < 0$.

$1\neq z_1\neq z_2\neq 1$.

racines cubiques de 1 $\neq 1$ $= \{z_1, z_2\} = \{e^{i2\pi/3}, e^{i4\pi/3}\}$.

donc comme $\mathrm{Im}\, z_1 < 0$,

$z_1 = e^{i4\pi/3}$

$z_2 = e^{i2\pi/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ (formule d'Euler) $= -\frac{1}{2} + i\frac{\sqrt3}{2}$

Donc $\boxed{\cos\frac{2\pi}{3} = -\frac{1}{2},\ \sin\frac{2\pi}{3} = \frac{\sqrt3}{2}.}$

$(\rho e^{i\varphi})^2 = re^{i\theta} \iff \begin{cases} \rho^2 = r \\ \text{et} \\ 2\varphi \equiv \theta \ [2\pi] \end{cases} \iff \begin{cases} \rho = \sqrt{r} \\ \varphi \equiv \frac{\theta}{2} \ [\pi] \end{cases}$ | $z = re^{i\theta}$

Les racines carrées de z sont $\sqrt{r}\, e^{i\theta/2}$ et $\sqrt{r}\, e^{i(\frac{\theta}{2}+\pi)}$

exercice 5.12

$$e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

1 - Les racines carrées de $e^{i2\pi/3}$ sont $e^{i\pi/3}$, $e^{i4\pi/3}$

$$z^2 = (a+ib)^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \iff \begin{cases} a^2 - b^2 = -\frac{1}{2} \\ 2ab = \frac{\sqrt{3}}{2} \end{cases}$$

$$\iff \begin{cases} a^2 - \frac{3}{16a^2} = -\frac{1}{2} \\ a \neq 0 \\ b = \frac{\sqrt{3}}{4a} \end{cases}$$

$$\iff \begin{cases} a^4 + \frac{1}{2}a^2 - \frac{3}{16} = 0 \\ a \neq 0 \\ b = \frac{\sqrt{3}}{4a}. \end{cases} \quad (*)$$

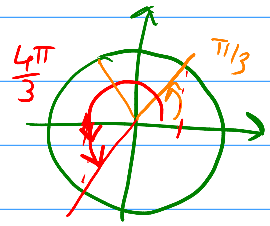
On résout $X^2 + \frac{1}{2}X - \frac{3}{16} = 0$ a pour solutions $\dfrac{-\frac{1}{2} \pm 1}{2} = \frac{1}{4}$ ou $-\frac{3}{4}$

$\Delta = \frac{1}{4} + \frac{3}{4} = 1$

donc $(*) \iff \begin{cases} a^2 = \frac{1}{4} \\ b = \frac{\sqrt{3}}{4a} \end{cases}$

$$\iff \begin{cases} a = \frac{1}{2} \\ b = \frac{\sqrt{3}}{2} \end{cases} \text{ ou } \begin{cases} a = -\frac{1}{2} \\ b = -\frac{\sqrt{3}}{2} \end{cases}$$

donc $\{e^{i\pi/3}, e^{i4\pi/3}\} = \{\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}$



donc $\cos\pi/3 = \frac{1}{2}$, $\sin\pi/3 = \frac{\sqrt{3}}{2}$

$e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$

$\pi/6 + \pi.$

2 - racines carrées de $e^{i\pi/3}$: $e^{i\pi/6}$, $e^{i7\pi/6}$.

$$z^2 = (a+ib)^2 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \iff \begin{cases} a^2 - b^2 = \frac{1}{2} \\ 2ab = \frac{\sqrt{3}}{2} \end{cases}$$

$$\iff \begin{cases} a^2 - \frac{3}{16a^2} = \frac{1}{2} \\ a \neq 0 \\ b = \frac{\sqrt{3}}{4a} \end{cases}$$

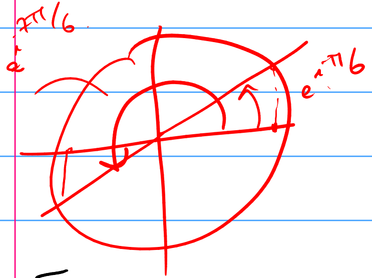
$$\iff \begin{cases} a^4 - \frac{1}{2}a^2 - \frac{3}{16} = 0 \\ a \neq 0 \\ b = \frac{\sqrt{3}}{4a} \end{cases} \qquad (\star\star)$$

$X^2 - \frac{1}{2}X - \frac{3}{16} = 0$ a pour solutions $\frac{\frac{1}{2} \pm 1}{2} = \begin{cases} -\frac{1}{4} \\ \text{ou} \\ \frac{3}{4} \end{cases}$

$\Delta = 1$

donc $(\star\star) \iff \begin{cases} a^2 = \frac{3}{4} \\ b = \frac{\sqrt{3}}{4a} \end{cases} \iff \begin{cases} a = \frac{\sqrt{3}}{2} \\ b = \frac{1}{2} \end{cases}$ ou $\begin{cases} a = -\frac{\sqrt{3}}{2} \\ b = -\frac{1}{2} \end{cases}$

donc $\boxed{e^{i\pi/6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}}$: $\begin{cases} \cos \pi/6 = \frac{\sqrt{3}}{2} \\ \sin \pi/6 = \frac{1}{2} \end{cases}$



[3 Calculer les racines carrés de $e^{i\pi/12}$ sous forme alg et trig et en déduire les valeurs de $\cos\frac{\pi}{12}$ et $\sin\frac{\pi}{12}$.].

[4. $e^{i\pi/24}$]
5. $e^{i\pi/48}$.

$\cos\theta$ $\sin\theta$].
$\cos\frac{\theta}{2}$ $\sin\frac{\theta}{2}$

S13 $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$. · Construire une bijection

$$\mathbb{Z}/n\mathbb{Z} \longrightarrow U_n$$

addition multiplication.

1- $f : \mathbb{Z} \to U_n$, $k \mapsto e^{i2k\pi/n}$. est bien définie car pour tout $k \in \mathbb{Z}$, $\left(e^{i2k\pi/n}\right)^n = e^{i2k\pi} = 1$ donc $e^{i2k\pi/n} \in U_n$.

2- si $k \equiv l\,[n]$, alors $\exists r \in \mathbb{Z}$ tq $l = k + nr$. Dans ce cas
$$e^{i2l\pi/n} = e^{i(2k\pi/n + 2r\pi)} = e^{i2k\pi/n}.$$

3- f est constante sur les classes d'équivalence de la relation de congruence modulo n donc elle induit $g : \mathbb{Z}/n\mathbb{Z} \to U_n$, $\bar{k} \mapsto e^{i2k\pi/n}$.

$U_n = \{e^{i\frac{2k\pi}{n}}, k = 0, \dots, n-1\}$, $\text{card}(U_n) = n$

4. $\text{card}\, \mathbb{Z}/n\mathbb{Z} = \text{card}\, U_n = n$.

g est surjective car son image $e^{i2k\pi/n}$, $k = 0, \dots, n-1$.
donc bijective.

définition.

5- si $x, y \in \mathbb{Z}/n\mathbb{Z}$, $x = \bar{k}$, $y = \bar{l}$, $x + y = \bar{k} + \bar{l} = \overline{k+l}$

donc $g(x+y) = e^{i\,2(k+l)\pi/n} = e^{i2k\pi/n} e^{i2l\pi/n}$

$e^{i2k\pi/n} = f(k) = g(\bar{k}) = g(x)$, $e^{i2l\pi/n} = f(l) = g(\bar{l}) = g(y)$

donc $\boxed{g(x+y) = g(x)g(y)}$

"g est isomorphisme de groupes entre $\mathbb{Z}/n\mathbb{Z}$ et U_n".

exercice 5.14 : Exprimer $\cos^3\theta = (\cos\theta)^3$ en fonction de $\cos(n\theta)$ où $\sin(n\theta)$ $n \in \mathbb{N}$. $(\cos\theta)^3 = \sum \lambda_n \cos(n\theta) + \sum \mu_n \sin n\theta$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$(\cos\theta)^3 = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^3 = \frac{1}{8}\left(e^{i3\theta} + 3\underbrace{e^{i2\theta}e^{-i\theta}}_{e^{i\theta}} + 3\underbrace{e^{i\theta}e^{-i2\theta}}_{e^{-i\theta}} + e^{-i3\theta}\right)$$

$$= \frac{1}{8}\left(e^{i3\theta} + e^{-i3\theta} + 3\left(e^{i\theta} + e^{-i\theta}\right)\right)$$

$\cos(2\theta) = 2\cos^2\theta - 1.$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(2a) = \cos^2 a - \sin^2 a$ $(+\cos^2 a - \cos^2 a)$

$= 2\cos^2 a - 1$

$\cos^2 a = \dfrac{\cos(2a)+1}{2}$

$\;= \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right)$ $\quad \overset{\shortparallel}{2\cos(3\theta)} \quad \overset{\shortparallel}{2\cos\theta}$

$$\sin^3\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3 = \frac{1}{8i^3}\left(e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}\right)$$

$$= \frac{1}{8\cdot(-i)}\left(e^{i3\theta} - e^{-i3\theta} - 3(e^{i\theta}-e^{-i\theta})\right)$$

$$= -\frac{1}{8i}\left(2i\sin(3\theta) - 3\cdot 2i\sin(\theta)\right)$$

$$= -\frac{1}{4}\sin(3\theta) + \frac{3}{4}\sin(\theta)$$

Calculs : $n \geqslant 0$

$$\sum_{k=0}^{n}\cos(kx) = \mathrm{Re}\Big(\underbrace{\sum_{k=0}^{n} e^{ikx}}_{\text{progression géométrique de raison } e^{ix}}\Big)$$

$$= \begin{cases} \mathrm{Re}(n+1) = n+1 & \text{si } x \equiv 0\ [2\pi] \\ \mathrm{Re}\left(\dfrac{1-e^{i(n+1)x}}{1-e^{ix}}\right) & \text{si } x \not\equiv 0\ [2\pi] \end{cases}$$

factorisation par l'angle moitié.

$$\frac{1-e^{i(n+1)x}}{1-e^{ix}} = \frac{e^{i\frac{(n+1)x}{2}}\left(e^{-i\frac{(n+1)x}{2}} - e^{i\frac{(n+1)x}{2}}\right)}{e^{i\frac{x}{2}}\left(e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}\right)}$$

$\frac{e^{i\frac{(n+1)x}{2}}}{e^{i\frac{x}{2}}} = e^{i\frac{nx}{2}}$

$e^{-i\frac{(n+1)x}{2}} - e^{i\frac{(n+1)x}{2}} = 2i\,\mathrm{Im}\left(e^{-i\frac{(n+1)x}{2}}\right) = -2i\sin\left(\frac{(n+1)x}{2}\right)$

$e^{-i\frac{x}{2}} - e^{i\frac{x}{2}} = 2i\,\mathrm{Im}\left(e^{-i\frac{x}{2}}\right) = 2i\sin\left(-\frac{x}{2}\right) = -2i\sin\left(\frac{x}{2}\right)$

$$= e^{i\frac{nx}{2}}\cdot\frac{\sin\left(\frac{(n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

$\dfrac{\sin\left(\frac{(n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \in \mathbb{R}$

$$\boxed{\text{donc } \sum_{k=0}^{n}\cos(kx) = \cos\left(\frac{nx}{2}\right)\cdot\frac{\sin\left(\frac{(n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}}$$

Même méthode $\displaystyle\sum_{k=0}^{n}\sin(kx)$

$$\sum_{k=0}^{n} (-1)^k 2^{-k} \cos(kx) = \mathrm{Re}\left(\sum_{k=0}^{n} \underbrace{(-1)^k 2^{-k} e^{ikx}}_{\left(-\frac{e^{ix}}{2}\right)^k} \right).$$

$$\sum_{k=0}^{n} \binom{n}{k} \sin(kx) = \mathrm{Im}\left(\sum_{k=0}^{n} \binom{n}{k} e^{ikx} \right)$$

$$= \mathrm{Im}\left((1+e^{ix})^n \right).$$

binôme.

$$1+e^{ix} = e^{i\frac{x}{2}}\left(e^{-i\frac{x}{2}} + e^{i\frac{x}{2}}\right)$$

$$= e^{i\frac{x}{2}} \cdot 2\cos\left(\frac{x}{2}\right)$$

$$= \mathrm{Im}\left(e^{i\frac{nx}{2}} \cdot 2^n \cos^n\left(\frac{x}{2}\right)\right)$$

$$= \sin\left(\frac{nx}{2}\right) 2^n \cos^n\left(\frac{x}{2}\right).$$

Résoudre $\begin{cases} z \neq 1 \\ 27(z-1)^6 + (z+1)^6 = 0. \end{cases}$ inconnue. $z \in \mathbb{C}$

$$\Leftrightarrow \begin{cases} \dfrac{(z+1)^6}{(z-1)^6} = -27. \\ z \neq 1. \end{cases}$$

$$\Leftrightarrow \begin{cases} Z = \dfrac{z+1}{z-1} \\ Z^6 = -27 \\ z \neq 1 \end{cases}$$

$$Z^6 = -27 = -3^3 = 3^3 e^{i\pi} \Leftrightarrow Z = (3^3)^{1/6} e^{i\frac{\pi+2k\pi}{6}}, \; k = 0, \ldots, 5.$$

$$= \sqrt{3}\, e^{i\frac{\pi+2k\pi}{6}} \quad k = 0, \ldots, 5.$$

$$\Leftrightarrow \begin{cases} \dfrac{z+1}{z-1} = e^{i\frac{\pi+2k\pi}{6}} \quad k = 0, \ldots, 5. \\ z \neq 1 \end{cases}$$

$$\begin{cases} \dfrac{z+1}{z-1} = a \\ z \neq 1 \end{cases} \Leftrightarrow \begin{cases} z+1 = a(z-1) \\ z \neq 1 \end{cases} \Leftrightarrow \begin{cases} z(a-1) = a+1 \\ z \neq 1 \end{cases}$$

$$\Leftrightarrow z = \frac{a+1}{a-1}.$$

$$\Leftrightarrow \quad z = \frac{e^{i\frac{\pi+2k\pi}{6}}+1}{e^{i\frac{\pi+2k\pi}{6}}-1} \quad k \in \{0, \dots, 5\}.$$

si $k=0$,
$$\frac{e^{i\pi/6}+1}{e^{i\pi/6}-1} = \frac{\cancel{e^{i\pi/12}}\left(e^{i\pi/12}+e^{-i\pi/12}\right)}{\cancel{e^{i\pi/12}}\left(e^{i\pi/12}-e^{-i\pi/12}\right)}$$
$$= -i\,\frac{\cos(\pi/12)}{\sin(\pi/12)}$$