TD15 - Vendredi 4 décembre 2020

ex 5.6 : 2 façons : ① $(a+ib)^2 = z$ $\Rightarrow$ équations pour a et b

$$\begin{cases} a^2-b^2 = \mathrm{Re}(z) \\ 2ab = \mathrm{Im}(z) \end{cases}$$

② $z = |z| e^{i\,\mathrm{Arg}(z)}$ les racines carrés de z sont $\pm\sqrt{|z|}\, e^{i\,\mathrm{Arg}(z)/2}$.

- $z_1 = -1$ Les racines carrées sont $\pm i$.

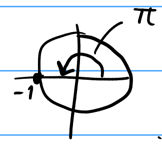
Soit $z = a+ib \in \mathbb{C}$, $a, b \in \mathbb{R}$

$$z^2 = -1 \Leftrightarrow \begin{cases} a^2-b^2 = -1 \\ \cancel{2}ab = 0 \end{cases} \Leftrightarrow \left(\begin{cases} a^2-b^2 = -1 \\ a = 0 \end{cases} \text{ ou } \begin{cases} a^2-b^2=-1 \\ b = 0 \end{cases} \right)$$

$$\Leftrightarrow \left(\begin{cases} -b^2 = -1 \\ a = 0 \end{cases} \text{ ou } \begin{cases} a^2 = -1 \\ b = 0 \end{cases} \right)$$

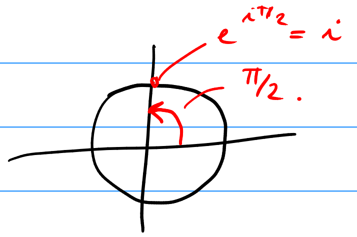
$$\Leftrightarrow \begin{cases} b = \pm 1 \\ a = 0 \end{cases}$$

$$\Leftrightarrow z = \pm i.$$



- **2ème méthode :** $-1 = e^{i\pi}$

donc les racines carrées de -1 sont $\pm e^{i\pi/2} = \pm i$.



$z_2 = 1+i$ Soit $z = a+ib \in \mathbb{C}$, $a, b \in \mathbb{R}$

$$(a^2-b^2)+2iab = z^2 = z_2 \Leftrightarrow \begin{cases} a^2-b^2 = 1 \\ 2ab = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a^2-b^2 = 1 \\ a \neq 0 \\ b = \dfrac{1}{2a} \end{cases}$$

$$\Leftrightarrow \begin{cases} a^2 - \dfrac{1}{4a^2} = 1 \\ a \neq 0 \\ b = \dfrac{1}{2a} \end{cases}$$

$$\Leftrightarrow \begin{cases} a^4 - a^2 - \dfrac{1}{4} = 0 \\ a \neq 0 \\ b = \dfrac{1}{2a} \end{cases} \quad (*)$$

On résout $a^4 - a^2 - \frac{1}{4} = 0$.

$$a^4 - a^2 - \frac{1}{4} = 0 \Leftrightarrow \begin{cases} X = a^2 \\ X^2 - X - \dfrac{1}{4} = 0 \end{cases} \Leftrightarrow \begin{cases} X = a^2 \\ X = \dfrac{1 \pm \sqrt{2}}{2} \end{cases}$$

$$\left[\ \Delta = 1 + 1 = 2\ \right]$$

$$\Leftrightarrow \begin{cases} X = a^2 \\ X = \dfrac{1+\sqrt{2}}{2} \end{cases}$$

$1 = \sqrt{1} < \sqrt{2} \simeq 1,4$

donc $1 - \sqrt{2} < 0$

$$\Leftrightarrow \begin{cases} a = \pm \sqrt{\dfrac{1+\sqrt{2}}{2}} \\ X = a^2. \end{cases}$$

$$(*) \Leftrightarrow \begin{cases} a = \sqrt{\dfrac{1+\sqrt{2}}{2}} \\ b = \dfrac{1}{2} \dfrac{\sqrt{2}}{\sqrt{1+\sqrt{2}}} \end{cases} \text{ ou } \begin{cases} a = -\sqrt{\dfrac{1+\sqrt{2}}{2}} \\ b = -\dfrac{1}{2} \dfrac{\sqrt{2}}{\sqrt{1+\sqrt{2}}} \end{cases}$$

Les racines carrées de $1+i$ sont $\pm \left(\sqrt{\dfrac{1+\sqrt{2}}{2}} + i \dfrac{1}{2} \dfrac{\sqrt{2}}{\sqrt{1+\sqrt{2}}} \right)$

$$\frac{1}{2} \frac{\sqrt{2}}{\sqrt{1+\sqrt{2}}} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1+\sqrt{2}}} = \frac{1}{\sqrt{2+2\sqrt{2}}}.$$

<u>2ème méthode</u> : $1+i = \sqrt{2}\, e^{i\pi/4}$

$\sqrt{\sqrt{2}} = 2^{1/4}$

$\rightsquigarrow$ racines carrées de $1+i$ sont $\pm 2^{1/4} e^{i\pi/8}$.

$$= \pm 2^{1/4} \left(\cos\frac{\pi}{8} + i \sin\frac{\pi}{8} \right)$$

On ne connait pas $\cos\left(\frac{\pi}{8}\right)$.

On trouve $\cos\frac{\pi}{8} = \dfrac{1}{2^{1/4}} \sqrt{\dfrac{1+\sqrt{2}}{2}}$ et $\sin\frac{\pi}{8} = \dfrac{1}{2^{1/4}} \dfrac{1}{\sqrt{2+2\sqrt{2}}}$.

$$= \frac{1}{\sqrt{\sqrt{2}}} \sqrt{\frac{1+\sqrt{2}}{2}} \qquad = \frac{1}{\sqrt{4+2\sqrt{2}}}$$

$$= \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$$

$z_3 = -1-i = -z_2$

Soit $z \in \mathbb{C}$. $z^2 = z_3 \iff z^2 = -z_2$

$\iff -z^2 = z_2$

$\iff (iz)^2 = z_2$

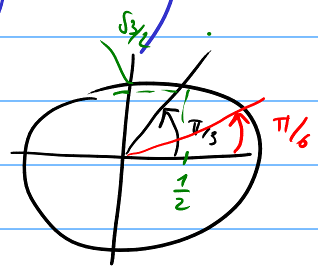
$\iff iz = \underbrace{\pm\left(\sqrt{\frac{1+\sqrt{2}}{2}} + i\sqrt{\frac{1}{2+2\sqrt{2}}}\right)}$ $\quad \left[\frac{1}{i} = -i\right]$

$\iff z = -i \cdot \underbrace{\qquad}$

$\iff z = \pm\left(\sqrt{\frac{1}{2+2\sqrt{2}}} - i\sqrt{\frac{1+\sqrt{2}}{2}}\right)$

$z_4 = 1+i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 2e^{i\pi/3}$



donc les racines de z^4 sont $\pm\sqrt{2}e^{i\pi/6} = \pm\sqrt{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$z_5 = 3-4i$. $\qquad |z_5|^2 = 25$

$= 5\cdot\left(\frac{3}{5} - \frac{4}{5}i\right)$: pas facile d'écrire sous forme trigonométrique.

Soit $z \in \mathbb{C}$, $z = a+ib$, $a, b \in \mathbb{R}$.

$$z^2 = z_5 \iff \begin{cases} a^2 - b^2 = 3 \\ 2ab = -4. \end{cases} \iff \begin{cases} a^2 - \frac{4}{a^2} = 3 \\ a \neq 0 \\ b = \frac{-4}{2a} = \frac{-2}{a} \end{cases}$$

$$\iff \begin{cases} a^4 - 3a^2 - 4 = 0 \\ a \neq 0 \\ b = -\frac{2}{a} \end{cases} \qquad \Delta = 25$$

$$\Leftrightarrow \begin{cases} a^2 = \dfrac{3 \pm 5}{2} = -1 \text{ ou } 4 \\ a \neq 0 \\ b = -\dfrac{2}{a} \end{cases}$$

$$\Leftrightarrow \begin{cases} a^2 = 4 \\ b = -\dfrac{2}{a} \end{cases}$$

$$\Leftrightarrow \begin{cases} a = 2 \\ b = -1 \end{cases} \text{ ou } \begin{cases} a = -2 \\ b = 1 \end{cases}$$

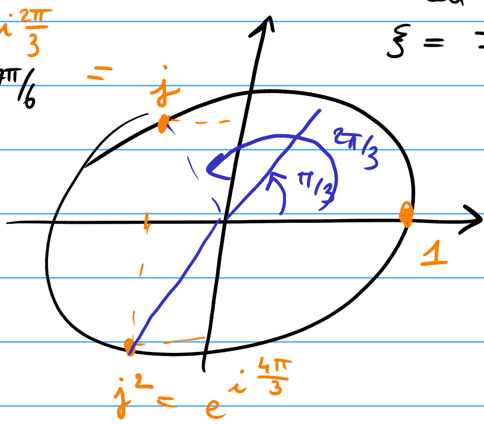
donc les $\sqrt{\cdot}$ de $3-4i$ sont $\boxed{\pm(2-i)}$.

5.7 $z^3 = i = e^{i\pi/2}$

$\xi \in \mathbb{C}$

$$\xi^3 = 1 \Leftrightarrow \begin{cases} \xi = 1 \\ \text{ou} \\ \xi = \dfrac{-1+i\sqrt{3}}{2} = j \\ \text{ou} \\ \xi = \dfrac{-1-i\sqrt{3}}{2} = j^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} z = e^{i\pi/6} \\ \text{ou} \\ z = j e^{i\pi/6} = e^{i\left(\frac{2\pi}{3}+\frac{\pi}{6}\right)} = e^{i\,7\pi/6} \\ \text{ou} \\ z = j^2 e^{i\pi/6} = e^{i\,9\pi/6} = e^{i\,3\pi/2} = e^{-i\pi/2} \end{cases}$$



Soit $\xi \in \mathbb{C}$.

Plus généralement résoudre $z^n = \xi = \rho e^{i\varphi}$ d'inconnue z :

$z = re^{i\theta}$

$$\Leftrightarrow r^n e^{in\theta} = \rho e^{i\varphi}$$

$$\Leftrightarrow \begin{cases} r^n = \rho \\ n\theta \equiv \varphi \ [2\pi]. \end{cases}$$

$$\Leftrightarrow \begin{cases} r = \rho^{1/n} \\ \theta = \dfrac{\varphi}{n} \left[\dfrac{2\pi}{n}\right] \end{cases}$$

$$z^3 = \left(re^{i\theta}\right)^3 = e^{i\pi/2} \iff \begin{cases} r^3 = 1 \\ 3\theta \equiv \frac{\pi}{2} \quad [2\pi] \end{cases}$$

$$\iff \begin{cases} r = 1 \\ \theta \equiv \frac{\pi}{6} \quad [\frac{2\pi}{3}] \end{cases}.$$

$$\iff \begin{cases} r = 1 \\ \theta \equiv \frac{\pi}{6} [2\pi] \text{ ou } \theta = \frac{\pi}{6} + \frac{2\pi}{3} [2\pi] \text{ ou } \theta = \frac{\pi}{6} + \frac{4\pi}{3} [2\pi]. \end{cases}$$

<u>ex. 5.8</u> : $z^2 + (2-3i)z - 5 - i = 0$

$$\Delta = (2-3i)^2 - 4\cdot(-5-i)$$

$$= 4 - 9 - 12i + 20 + 4i$$

$$= 15 - 8i$$

<u>solution</u> : $\dfrac{-(2-3i) \pm \sqrt{15-8i}}{2}$

Soit $z = a+ib$ $\mathbb{C}$, $a, b \in \mathbb{R}$

$$z^2 = 15 - 8i \iff \begin{cases} a^2 - b^2 = 15 \\ 2ab = -8 \end{cases} \iff \begin{cases} a^2 - \frac{16}{a^2} = 15 \\ a \neq 0 \\ b = -\frac{4}{a} \end{cases}$$

$$\Delta = 225 + \overbrace{4\cdot 16}^{64}$$
$$= 289$$
$$= 17^2$$

$$\iff \begin{cases} a^4 - 15a^2 - 16 = 0 \\ a \neq 0 \\ b = -\frac{1}{a} \end{cases}$$

$$\iff \begin{cases} a^2 = \frac{15 \pm 17}{2} \\ a \neq 0 \\ b = -\frac{1}{a} \end{cases}$$

$$\iff \begin{cases} a = 16 \\ b = -\frac{1}{16} \end{cases} \text{ ou } \begin{cases} a = -16 \\ b = \frac{1}{16} \end{cases}$$

donc les sol sont $\dfrac{-(2-3i) \pm \left(16 - \frac{1}{16} \cdot i\right)}{2}$

$$= \begin{cases} \frac{1}{2}\left(14 + \left(3 - \frac{1}{16}\right) i\right) \\ \text{ou} \\ \frac{1}{2}\left(-18 + \left(3 + \frac{1}{16}\right) i\right). \end{cases}$$

5.9. $\left(\underbrace{\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)}_{\text{formule d'Euler}}\right)^{2021} = \sum_{k=0}^{2021} \binom{2021}{k} \cos\left(\frac{2\pi}{5}\right)^k \left(i \sin\left(\frac{2\pi}{5}\right)\right)^{2021-k}$

(formule du binôme).

On ne sait pas simplifier.

$e^{i \cdot \frac{2\pi}{5}} = \xi$

donc $\xi^{2021} = e^{i\, 2021 \cdot \frac{2\pi}{5}} = e^{i \frac{2021}{5} \cdot 2\pi}$.

$2021 = 5 \cdot 404 + 1.$

$$\begin{array}{r|l} 2021 & 5 \\ \hline 20 & 404 \\ 021 & \\ 1 & \end{array}$$

$= e^{i \frac{5 \cdot 404 + 1}{5} \cdot 2\pi}$

$= e^{i\, 404 \cdot 2\pi + \frac{2\pi}{5} i}$

$= e^{i\, 2\pi/5}$

$= \xi = \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)$

$\frac{2021}{5} \cdot 2\pi \equiv \frac{5 \cdot 404 + 1}{5} 2\pi \; [2\pi]$

$\equiv \cancel{404 \cdot 2\pi} + \frac{2\pi}{5} \; [2\pi]$

$\equiv \frac{2\pi}{5} \; [2\pi].$

$\xi^{2021} = \xi.$

$\xi^5 = \left(e^{2i\pi/5}\right)^5 = e^{2i\pi} = 1.$

donc $\xi^{2021} = \xi^{5 \cdot 404 + 1} = \left(\xi^5\right)^{404} \cdot \xi$

$= \xi.$

5.10. Soit $z \in \mathbb{C}$

$$(z-1)(\underbrace{z^{n-1}+\dots+1}_{\text{// si } z\neq 1 \quad \frac{1-z^n}{1-z}}) = (z-1)\left(\sum_{k=0}^{n-1} z^k\right)$$

$$= \sum_{k=0}^{n-1} z^{k+1} - \sum_{k=0}^{n-1} z^k$$

$$= \sum_{k=1}^{n} z^k - \sum_{k=0}^{n-1} z^k$$

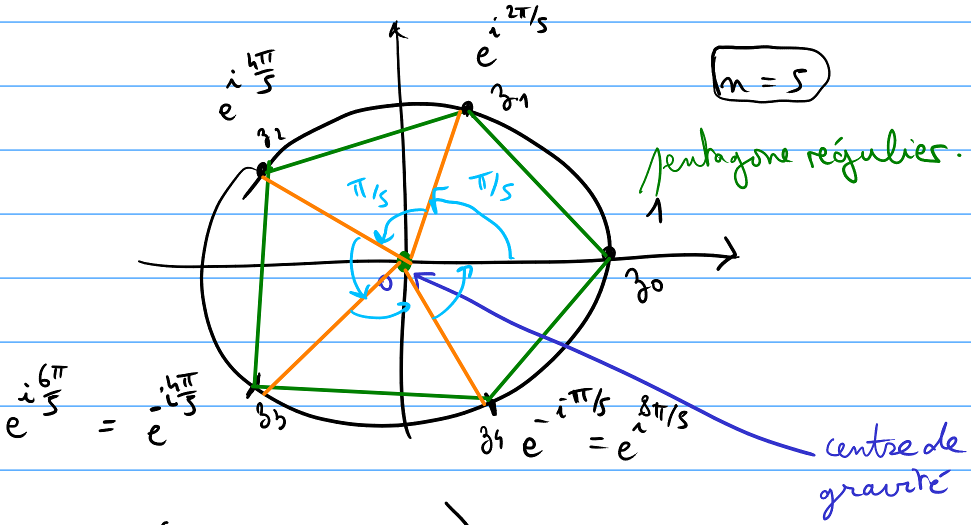
$$= z^n - 1.$$

2. On a $\begin{cases} z^n - 1 = 0 \\ z \neq 1 \end{cases} \Leftrightarrow \begin{cases} (z-1)(z^{n-1}+\dots+1) = 0 \\ z-1 \neq 0 \end{cases}$

$\Leftrightarrow z^{n-1} + \dots + 1 = 0.$

3. $z = e^{i\frac{2\pi}{n}} \neq 1$. $z^k = e^{i\frac{2k\pi}{n}}$ → pour $k = 0, \dots, n-1$, on a toutes les racines n-ièmes de 1 $\quad n \geq 2$ donc $0 < \frac{2\pi}{n} < 2\pi$

et par 2., $\underbrace{1 + z + \dots + z^{n-1}}_{\sum_{k=0}^{n-1} z^k} = 0 \Leftarrow$ la somme des racines n-ièmes de 1 est nulle.



centre de gravité du pentagone. $= \frac{1}{5}(z_0+z_1+z_2+z_3+z_4) = 0$