TD11 - Vendredi 13 novembre 2020

Algorithme d'Euclide standard

$a, b \in \mathbb{Z}$.

$r_0 = a \quad , r_1 = b$

$r_0 = r_1 q_0 + r_2 \qquad q_0 \in \mathbb{Z}, \ 0 \leq r_2 < r_1$

$r_1 = r_2 q_1 + r_3 \qquad q_1 \in \mathbb{Z} \quad 0 \leq r_3 < r_2$

$\vdots$

$r_{n-1} = r_n q_{n-1} + r_{n+1} \qquad r_{n+1} \neq 0 \longrightarrow r_{n+1} = \text{pgcd}(a, b)$

$r_n = r_{n+1} q_n + \cancel{r_{n+2}}_{=0}$ fin de l'algorithme

Relation de Bezout : Soit $a, b \in \mathbb{Z}$. $\text{pgcd}(a,b) = 1$, $\Leftrightarrow$ $\exists u, v \in \mathbb{Z}$ tq $au + bv = 1$.

Soit $a, b \in \mathbb{Z}$. $\text{pgcd}(a, b) = d$ $\Rightarrow \exists \ u, v \in \mathbb{Z}$ tq
$d \in \mathbb{Z}$ $\qquad au + bv = d$.

Algorithme d'Euclide étendue

$a, b \in \mathbb{Z} \qquad r_0 = a, r_1 = b$.

$r_0 = \boxed{u_0 a + v_0 b}$

$r_1 = \boxed{u_1 a + v_1 b}$

$r_2 = u_2 a + v_2 b$

$r_2 = u_0 a + v_0 b - q_0(u_1 a + v_1 b)$
$= a(\underbrace{u_0 - q_0 u_1}_{u_2}) + b(\underbrace{v_0 - q_0 v_1}_{v_2})$

		$r_0 = a$	$r_1 = b$
	$\rightarrow r_0 = a$	$u_0 = 1$	$v_0 = 0$
	$\rightarrow r_1 = b$	$u_1 = 0$	$v_1 = 1$
$r_0 = r_1 q_0 + r_2$	$\rightarrow r_2 = r_0 - r_1 q_0$	$u_2 = u_0 - q_0 u_1$	$v_2 = v_0 - q_0 v_1$.
$r_1 = r_2 q_1 + r_3$	$r_3 = r_1 - r_2 q_1$	$u_3 = u_1 - q_1 u_2$	$v_3 = v_1 - q_1 v_2$
$\vdots$			
$r_{n-1} = r_n q_{n-1} + r_{n+1}$, $r_{n+1} \neq 0$	$r_{n+1} = r_{n-1} - r_n q_{n-1}$	u_{n+1}	v_{n+1}
$r_n = r_{n+1} q_n + 0$			

$\underbrace{u_{n+1}}_{u} a + \underbrace{v_{n+1}}_{v} b = r_{n+1} = \text{pgcd}(a,b)$

Rq si $\underset{r_0}{a} < \underset{r_1}{b}$ $\qquad r_0 = 0 \cdot r_1 + \underset{r_2}{r_0}$

on trouve r_1

$r_2 = r_0$

$r_1 = \underset{r_0}{r_2} \cdot q_1 + r_3$

u, v pas uniques : $a = 4,\ 4 = 3$ $\quad pgcd(3,4) = 1$.

$$4 + (-1)\cdot 3 = 1. \qquad u = 1,\ v = -1$$

$$4\cdot 4 + (-5)\cdot 3 = 1 \qquad u = 4,\ v = -5.$$

$\vdots$

Feuille 4 :

4.1] 1. $a = 13,\ b = 17$

		$r_0 = a = 13$	$r_1 = b = 17$
	$r_0 = a = 13$	$u_0 = 1$	$v_0 = 0$
	$r_1 = b = 17$	$u_1 = 0$	$v_1 = 1$
$13 = 17\times 0 + 13$	$13 = 13 - 17\times 0$	$u_2 = 1 = u_0 - 0\cdot u_1$	$v_2 = 0 = v_0 - 0\cdot v_1$
$17 = 13\times 1 + 4$	$4 = 17 - 13$	$u_3 = u_1 - u_2 = -1$	$v_3 = v_1 - v_2 = 1$
$13 = 4\times 3 + 1$	$1 = 13 - 4\times 3$	$u_4 = u_2 - 3\cdot u_3 = 1 + 3 = 4$	$v_4 = v_2 - 3\cdot v_3 = -3$
$4 = 1\cdot 4 + 0$ (reste nul)			

$$pgcd(13, 17) = 1 \qquad 4\cdot 13 - 3\cdot 17 = 1$$

$$(52 - 51) = 1$$

$r_0 = a = 32,\ r_1 = b = 27$

		a	b
	$r_0 = a = 32$	1	0
	$r_1 = b = 27$	0	1
$32 = 27\times 1 + 5$	$5 = 32 - 27$	1	-1
$27 = 5\times 5 + 2$	$2 = 27 - 5\times 5$	-5	$1 - 5\times(-1) = 6$
$5 = 2\times 2 + 1$	$1 = 5 - 2\times 2$	$1 - 2\times(-5) = 11$	$-1 - 2\times 6 = -13$
$2 = 1\times 2 + 0$			

$$pgcd(32, 27) = 1 = 11\cdot 32 - 13\cdot 27$$

$11\cdot 32 = 352$

$$\begin{array}{r} 27 \\ 13 \\ \hline 81 \\ 270 \\ \hline \boxed{351} \end{array}$$

$r_0 = a = 98$; $r_1 = b = 18$

		a	b
	$\to r_0 = 98$	$u_0 = 1$	$v_0 = 0$
	$\to r_1 = 18$	0	1
$98 = 18 \times 5 + 8$	$8 = 98 - 18 \cdot 5$	1	-5
$18 = 8 \times 2 + 2$	$2 = 18 - 8 \times 2$	-2	$1 - 2 \cdot (-5) = 11$
$8 = 2 \cdot 4 + 0$			

$$\text{pgcd}(98, 18) = 2 = -2 \cdot 98 + 11 \cdot 18$$

$$2 \cdot 98 = 196$$
$$11 \cdot 18 = 198 .$$

4. $a = n + 1$ $\quad b = n$

$$\begin{cases} (n+1) + (-1) \cdot n = 1 \\ a - b = 1 \end{cases} \quad \text{donc} \quad \text{pgcd}(n+1, n) = 1 .$$

<u>exo 4.2</u> : 1-

$$2975 = 1995 + 980$$
$$1995 = 980 \times 2 + 35$$
$$980 = 35 \times 28 + 0$$

$$\begin{array}{r|l} 980 & 35 \\ \hline 700 & 28 \\ 280 & \end{array}$$

donc $\text{pgcd}(1995, 2975) = 35$.

2- $\begin{cases} \exists q \in \mathbb{Z} \text{ tq } & 2003 = n \cdot q + 8 \\ \exists q' \in \mathbb{Z} \text{ tq } & 3002 = n \cdot q' + 27 . \end{cases}$ $\quad 0 \leq 8 < n$, $\quad 0 \leq 27 < n$

donc

$$nq = 2003 - 8 = 1995$$
$$nq' = 3002 - 27 = 2975$$

donc $n \,/\, 1995$ et $n \,/\, 2975$

donc $n \,/\, \text{pgcd}(1995, 2975) = 35$

On a $\begin{cases} n \mid 35 \\ n < 27 \end{cases} \Rightarrow \underline{n = 35}.$

diviseurs de $\underset{5\times 7}{35}$: 1, 5, 7, 35.

exo 43 ; 1. a. 3, 5.

Scores atteignables avec 3 et 5 : 3, 6, 9 ; 5, 8

$+12 \cdot 3 - 7 \cdot 5 = +1$ ($12\cdot 3 = 36$, $7\cdot 5 = 35$)

b. $3u + 5v = 1$

$\left[\begin{array}{ll} u = 2,\ v = -1 & (1) \\ u = -3,\ v = 2 & (2) \end{array}\right.$

c. initialisation : 9 est atteint

hérédité. Soit $n \geq 9$ tq $\exists\, \alpha, \beta \geq 0$, $3\alpha + 5\beta = n$.

On a $n+1 = 3\alpha + 5\beta + (3u + 5v)$
$= 3(\alpha + u) + 5(\beta + v)$

si $\beta > 0$, on prend (1) : $n+1 = 3(\underset{\geq 0}{\alpha + 2}) + 5(\underset{\geq 0}{\beta - 1})$

si $\beta = 0$, $n = 3\alpha$ et $n \geq 9 \Rightarrow \alpha \geq 3$. On prend (2) :

$n+1 = 3(\underset{\geq 0}{\alpha - 3}) + 5 \times \underset{\geq 0}{2}$

donc $n+1$ est atteignable.

Conclusion : $\forall n \geq 9$, $\exists\, \alpha, \beta \geq 0$, $3\alpha + 5\beta = n$.

2- Tous les scores ≥ 9,

3, 6, 9 ; 5, 8 et 7

3, 5, 6, 7, 8, 9.

pour le 20/11 . 4.4, 4.5