INVARIANTS AND WALL-CROSSING WITH JOYCE'S LIE ALGEBRA

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We follow Joyce's notation wherever possible so that the reader will hopefully find it easy to refer to his paper [Joy21]. The goal of the talk is to define Joyce's invariants and state his wall-crossing formula. We try to emphasise the role of Joyce's vertex algebra and Lie bracket.

1. Setup and Fragestellung

Start[1] with an abelian category $\mathcal{A}$ and consider a/the moduli stack $\mathcal{M}$ of objects in $\mathcal{A}$ where its $\mathbb{C}$-points are in one-to-one correspondence with (isomorphism classes of) objects in $\mathcal{A}$. Consider also the *projective-linear moduli of objects* $\mathcal{M}^{\mathrm{pl}} = \mathcal{M} /\!\!/ \mathbb{G}_m$ which is the rigidifcation of the moduli stack by the scaling automorphisms.

We choose a surjection $K_0(\mathcal{A}) \twoheadrightarrow K(\mathcal{A})$ onto a group (usually a lattice). Denote by $C(\mathcal{A})$ the image of $\mathcal{A}$ in $K(\mathcal{A})$. The moduli stacks $\mathcal{M}$ and $\mathcal{M}^{\mathrm{pl}}$ decompose into open and closed substacks $\mathcal{M}_\alpha$ and $\mathcal{M}_\alpha^{\mathrm{pl}}$ indexed by classes in $C(\mathcal{A})$:

$$\mathcal{M} = \bigsqcup_\alpha \mathcal{M}_\alpha \qquad \text{and} \qquad \mathcal{M}^{\mathrm{pl}} = \bigsqcup_\alpha \mathcal{M}_\alpha^{\mathrm{pl}}.$$

The following structures on $\mathcal{M}$ are required for the definition of Joyce's vertex algebra. We assume that we are additionally two morphisms of stacks: the first

$$\Phi\colon \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$$

should[2] be the direct sum of objects on $\mathbb{C}$-points and should endow $\mathcal{M}$ with the structure of a monoid stack, the second

$$\Psi\colon [\mathrm{pt}/\mathbb{C}^\times] \times \mathcal{M} \longrightarrow \mathcal{M}$$

should be the identity on the objects of $\mathbb{C}$-points, but should scale the automorphisms of the $\mathbb{C}$-points, and should endow the stack $\mathcal{M}$ with the action of the group stack $[\mathrm{pt}/\mathbb{C}^\times]$. We assume there is a perfect complext $\mathcal{E}^\bullet$ on

We are given quasi-smooth derived enhancements $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{M}}^{\mathrm{pl}}$ of $\mathcal{M}$ and $\mathcal{M}^{\mathrm{pl}}$, respectively. The 'quasi-smoothness' means that the derived enhancements induce perfect obstruction theories $E^\bullet = (\mathbb{L}_{\boldsymbol{\mathcal{M}}^{\mathrm{pl}}})|_{\mathcal{M}^{\mathrm{pl}}} \to \mathbb{L}_{\mathcal{M}^{\mathrm{pl}}}$. The reader is welcome to ignore the derived enhancements and just keep the obstruction theory in mind.

A *(weak) stability condition*[3] $(\tau, T, \leq) = \tau\colon C(\mathcal{A}) \to (T, \leq)$ on $\mathcal{A}$ is map τ from $C(\mathcal{A})$ to a partially ordered set $(T, \leq)$ such that for every $\alpha', \alpha'' \in C(\mathcal{A}), \alpha = \alpha' + \alpha''$ we have one of

$$\begin{aligned} &\tau(\alpha') < (\leq)\tau(\alpha) < (\leq)\tau(\alpha'') \text{ or} \\ &\tau(\alpha') = \tau(\alpha) = \tau(\alpha'') \text{ or} \\ &\tau(\alpha') > (\geq)\tau(\alpha) > (\geq)\tau(\alpha''). \end{aligned}$$

We define τ-stable, τ-semistable, τ-unstable, etc. objects in the usual way.

We only consider stability conditions τ such that the substacks

$$\mathcal{M}_\alpha^{\mathrm{pl}} \supset \mathcal{M}_\alpha^{\mathrm{ss}}(\tau) \supset \mathcal{M}_\alpha^{\mathrm{st}}(\tau).$$

of τ-semistable objects and of τ-stable objects are open.

[1] For the full details of the setup of the categories and moduli stacks consult [Joy21, Assumption 5.1].

[2] Whenever we use the word 'should' we meant that we are assuming something or assuming that extra data is given.

[3] For more details on stability conditions for abelian categories see [Joy21, §3.1]. For the assumptions on how stability conditions interact with the moduli theory see [Joy21, Assumptions 5.2]

Since the scaling $\mathbb{G}_m$ stabilisers have been rigidified away, the moduli stacks of stable objects $\mathcal{M}^{\mathrm{st}}_\alpha(\tau)$ are in fact schemes. Thus, whenever they are proper, restriction of the perfect obstruction theory yields the Behrend–Fantechi virtual fundamental class $[\mathcal{M}^{\mathrm{st}}_\alpha(\tau)]_{\mathrm{vir}} \in H_*(\mathcal{M}^{\mathrm{st}}_\alpha(\tau))$.

Aim. Define 'natural' classes $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}} \in H_*(\mathcal{M}^{\mathrm{pl}})$ for all α and τ, in particular when there are τ-strictly semistables. These should satisfy the following.

(i) If τ-semistable implies τ-stable, then $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}} = [\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{vir}}$ is the Behrend–Fantechi virtual fundamental class.

(ii) The classes satisfy a wall crossing formula, i.e. we can write invariants $[\mathcal{M}^{\mathrm{ss}}_\alpha(\widetilde{\tau})]_{\mathrm{inv}}$ for a different stability $\widetilde{\tau}$ condition in terms of invariants $[\mathcal{M}^{\mathrm{ss}}_{\alpha_i}(\tau)]_{\mathrm{inv}}$.

2. Definition of the classes and the wall-crossing formula

We use the category[4] of pairs $\bar{\mathcal{A}}$ and its moduli stack of objects $\bar{\mathcal{M}}$ to define the classes $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}}$.

We begin by defining the categories of pairs. Assume we are given exact functors $F_k\colon \mathcal{A} \to \mathrm{Vect}$.[5] The objects in the category $\bar{\mathcal{A}}$ are pairs (E,ρ) consisting of an object $E \in \mathcal{A}$ and a linear map $\rho\colon V \to F_k(E)$. Here we take $K(\bar{\mathcal{A}}) = K(\mathcal{A}) \times \mathbb{Z}$ and $[\![E,\rho]\!] = ([\![E]\!], \dim(V))$. As before we form the moduli stack of objects $\bar{\mathcal{M}}$ and projective-linear moduli stack of objects $\bar{\mathcal{M}}^{\mathrm{pl}}$. For every $(\alpha, d) \in K(\mathcal{A}) \times \mathbb{Z}$ we have the natural forgetful morphism

$$\Pi_{\mathcal{M}^{(\mathrm{pl})}_\alpha}\colon \bar{\mathcal{M}}^{(\mathrm{pl})}_{(\alpha,d)} \longrightarrow \mathcal{M}^{(\mathrm{pl})}_\alpha. \tag{1}$$

A stability condition τ on $\mathcal{A}$ induces a stability condition $\bar{\tau}^0_1$ on $\bar{\mathcal{A}}$. For the stability condition $\bar{\tau}^0_1$ an object $\bar{E} \in \bar{\mathcal{A}}$ is semistable if and only if it is stable. Moreover, if $\bar{E}$ has class $[\![\bar{E}]\!] = (\alpha, 1)$ and is $\bar{\tau}^0_1$-semistable, then the underlying object E in $\mathcal{A}$ is τ-semistable. Hence we have the moduli spaces $\mathcal{M}^{\mathrm{ss}}_{(\alpha,1)}(\bar{\tau}^0_1) = \mathcal{M}^{\mathrm{st}}_{(\alpha,1)}(\bar{\tau}^0_1)$ and the morphism (1) restricts to a morphism between stacks of semistables

$$\Pi_{\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)}\colon \bar{\mathcal{M}}^{\mathrm{ss}}_{(\alpha,1)}(\tau^0_1) \longrightarrow \mathcal{M}^{\mathrm{ss}}_\alpha(\tau).$$

The morphsims $\Pi_{\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)}$ are smooth. So, the quasi-smooth derived enhancements $\boldsymbol{\mathcal{M}}^{(\mathrm{pl})}$ pull-back quasi-smooth derived enhancements $\bar{\boldsymbol{\mathcal{M}}}^{(\mathrm{pl})}$.

This means that the moduli spaces $\bar{\mathcal{M}}^{\mathrm{ss}}_{(\alpha,1)}(\bar{\tau}^0_1)$ are in fact schemes and hence via the induced obstruction theories from $\bar{\boldsymbol{\mathcal{M}}}^{\mathrm{pl}}$ we can define the Behrend–Fantechi virtual fundamental classes $[\bar{\mathcal{M}}^{\mathrm{ss}}_{(\alpha,1)}(\bar{\tau}^0_1)]_{\mathrm{vir}}$.

Recall that $H_*(\mathcal{M}^{\mathrm{pl}})$ has a Lie bracket $\Big[-,-\Big]$ induced from the vertex algebra structure on $H_*(\mathcal{M})$.

Definition 2.1. The classes $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}}$ are defined recursively via the formula[6] in $H_*(\mathcal{M}^{\mathrm{pl}})$

$$(\Pi_{\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)})_*[\bar{\mathcal{M}}^{\mathrm{ss}}_{(\alpha,1)}(\tau)]_{\mathrm{vir}} \cap c_{\mathrm{top}}(\mathbb{T}_{\bar{\mathcal{M}}^{\mathrm{ss}}_\alpha(\bar{\tau}^0_1)/\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)}) = \\ = \sum_{\substack{\alpha=\sum_i \alpha_i \\ \tau(\alpha_i)=\tau(\alpha)}} \frac{(-1)^n \lambda_k(\alpha)}{n!} \Big[\Big[\dots\Big[\Big[[\mathcal{M}^{\mathrm{ss}}_{\alpha_1}(\tau)]_{\mathrm{inv}}, [\mathcal{M}^{\mathrm{ss}}_{\alpha_2}(\tau)]_{\mathrm{inv}}\Big]\Big], \dots\Big], [\mathcal{M}^{\mathrm{ss}}_{\alpha_n}(\tau)]_{\mathrm{inv}}\Big]$$

The sum is over all (ordered) $\alpha_1, \dots, \alpha_n \in C(\mathcal{A})$ satisfying the indicated condition.

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[4] For more details on auxiliary categories such as this see [Joy21, §5.2].

[5] A good example for these to keep in mind are the functors $E \mapsto H^0(E(k))$ for a coherent sheaf E on a polarised quasi-projective variety $(X, \mathcal{O}_X(1))$. We remark that these are only exact on a subcategory $\mathcal{A}_k \subseteq \mathcal{A}$ consisting of those sheaves E such that $E(k)$ is globally generated. By Serre's theorem for a fixed coherent sheaf E we can always find a k such that $E(k)$ is globally generated but this k is not uniform among all coherent sheaves. This is the reason we need to consider many k.

[6] Throughout there are implicit assumptions that guarantee that all sums are finite.

For any two stability conditions $\tau, \widetilde{\tau}$ there are combinatorially defined 'universal wall-crossing coefficients' for every $\alpha_1, \dots, \alpha_n \in K(\mathcal{A})$

$$\widetilde{U}(\alpha_1, \dots, \alpha_n; \tau, \widetilde{\tau}) \in \mathbb{Q}.$$

There is a notion of continuous 1-parameter family of stability conditions, which roughly speaking means that the behaviour of the stabilities along the families behaves as if the path where to cross finitely many walls in a wall and chamber structure.

Theorem 2.2 (Wall-crossing formula). *Let τ and $\widetilde{\tau}$ be stability conditions[7] for $\mathcal{A}$ connected by a continuous 1-parameter family of stability conditions.*

$$[\mathcal{M}^{\mathrm{ss}}_\alpha(\widetilde{\tau})]_{\mathrm{inv}} = \sum_{\alpha = \sum_i \alpha_i} \widetilde{U}(\alpha_1, \dots, \alpha_n; \tau, \widetilde{\tau}) \Big[\Big[\dots\Big[\Big[[\mathcal{M}^{\mathrm{ss}}_{\alpha_1}(\tau)]_{\mathrm{inv}}, [\mathcal{M}^{\mathrm{ss}}_{\alpha_2}(\tau)]_{\mathrm{inv}}\Big]\Big], \dots\Big], [\mathcal{M}^{\mathrm{ss}}_{\alpha_n}(\tau)]_{\mathrm{inv}}\Big]$$

The sum is over all (ordered) $\alpha_1, \dots, \alpha_n \in C(\mathcal{A})$ satisfying the indicated condition.

3. Lie brackets and virtual localisation

The key observation for the appearance of the the Lie bracket in these relations is the following (easy) proposition.

Proposition 3.1. *Let X be proper algebraic space with $\mathbb{G}_m$-equivariant obstruction theory $\mathcal{F}^\bullet \to \mathbb{L}_X$.*

Let X_a be the fixed components of X and let $\mathcal{N}^\bullet_a$ be the virtual conormal bundle. We have the virtual classes $[X_a]_{\mathrm{vir}} \in H_*(X_a)$ and Euler classes $e(\mathcal{N}^\bullet_a) \in H_*(X_a)[z^{\pm 1}]$.

Let $f\colon X \to \mathcal{Y}$ be $\mathbb{G}_m$-equivariant morphism of Artin stacks with trivial $\mathbb{G}_m$-action on $\mathcal{Y}$. Then for all $\eta \in H^*_{\mathbb{G}_m}(X)$ we have

$$\sum_{X_a \in \pi_0(X^{\mathbb{G}_m})} (-1)^{\mathrm{rank}(\mathcal{N}^\bullet_a)} \operatorname{Res}_z \left[(f \circ i_a)_*([X_a]_{\mathrm{vir}} \cap (e(\mathcal{N}^\bullet_a)^{-1} \cup i_a^*(\eta)))\right] = 0 \tag{2}$$

Proof. This follows from the localisation formula for virtual fundamental classes. Indeed we have

$$\sum_{X_a \in \pi_0(X^{\mathbb{G}_m})} (-1)^{\mathrm{rank}(\mathcal{N}^\bullet_a)} (f \circ i_a)_*([X_a]_{\mathrm{vir}} \cap (e(\mathcal{N}^\bullet_a)^{-1} \cup i_a^*(\eta))) = f_*([X]_{\mathrm{vir}} \cap \eta)$$

We deduce the proposition by taking residues and noticing that $f_*([X]_{\mathrm{vir}} \cap \eta) \in H^{\mathbb{G}_m}_*(Y) = H_*(Y)[z]$ doesn't have a pole in z. □

Let Y denote the state-field correspondence for Joyce's vertex algebra on $H_*(\mathcal{M})$. It is given by

$$Y(u, z)v = \pm \sum_{i,j \geq 0} z^{*-i+j} (\Phi \circ (\Psi \times \mathrm{id}))_*(t^j \boxtimes ((u \boxtimes v) \cap c_i(\mathcal{E}^\bullet \oplus \sigma(\mathcal{E}^\bullet)^\vee))). \tag{3}$$

Recall that the Lie bracket $\Big[-, -\Big]$ can be expressed as a residue: for every $u, v \in H_*(\mathcal{M}^{\mathrm{pl}})$ we have

$$\Big[u, v\Big] = \operatorname{Res}_z(Y(u, z)v) \tag{4}$$

We explain the vague idea to obtain relations involving the classes

$$[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{inv}}, [\mathcal{M}^{\mathrm{ss}}_\alpha(\widetilde{\tau})]_{\mathrm{inv}}, \Big[[\mathcal{M}^{\mathrm{ss}}_{\alpha_1}(\tau)]_{\mathrm{inv}}, [\mathcal{M}^{\mathrm{ss}}_{\alpha_2}(\tau)]_{\mathrm{inv}}\Big]. \tag{5}$$

One constructs a 'master space'[8] X with a $\mathbb{G}_m$-action and a $\mathbb{G}_m$-perfect obstruction theory and fixed components $X_\alpha \cong \mathcal{M}^{\mathrm{ss}}_\alpha(\tau)$, $\widetilde{X}_\alpha \cong \mathcal{M}^{\mathrm{ss}}_\alpha(\widetilde{\tau})$, and $X_{\alpha_1,\alpha_2} = \mathcal{M}^{\mathrm{ss}}_{\alpha_1}(\tau) \times \mathcal{M}^{\mathrm{ss}}_{\alpha_2}(\tau)$ such that the virtual conormal bundles $\mathcal{N}^\bullet$ of X_α, respectively $\widetilde{X}_\alpha$ restrict to the obstruction theories on $\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)$, respectively $\widetilde{X}_\alpha \cong \mathcal{M}^{\mathrm{ss}}_\alpha(\widetilde{\tau})$, and the virtual conormal bundle of the fixed component $X_{\alpha_1,\alpha_2} \cong \mathcal{M}^{\mathrm{ss}}_{\alpha_1}(\tau) \times \mathcal{M}^{\mathrm{ss}}_{\alpha_2}(\tau)$ is roughly given by $-(\mathcal{E}^\bullet \oplus \sigma(\mathcal{E}^\bullet)^\vee)|_{\mathcal{M}^{\mathrm{ss}}_{\alpha_1}(\tau) \times \mathcal{M}^{\mathrm{ss}}_{\alpha_2}(\tau)}$.

[7] For additional technical conditions we impose on stability conditions see [Joy21, Assumption 5.3].

[8] See also [KL13] (especially Appendix A) for a simple example of an instance of this strategy.

Now comparing with (2),(3), and (4) we hope it seems plausible to the reader that in this situation we can apply (2) to deduce a relation between the classes (5).

For example using this technique one can show the following.

Lemma 3.2 ([Joy21, §9.2])**.** *The classes* $[\mathcal{M}^{\mathrm{ss}}_{\alpha}(\tau)]_{\mathrm{inv}}$ *don't depend on the (implicit) choice of* k*.*

Proof sketch. Apply the master space strategy outlined above, by taking the master space to be moduli spaces $\acute{\mathcal{M}}^{\mathrm{ss}}_{\alpha}(\acute{\tau})$ of semistable objects in auxiliary category an $\acute{\mathcal{A}}$ whose objects are the data of an object$E \in \mathcal{A}$ and a diagram of vector spaces

$$\begin{array}{ccccc} & & V_1 & \xrightarrow{\rho_1} & F_{k_1}(E) \\ & \overset{\rho_3}{\nearrow} & & & \\ V_3 & & & & \\ & \underset{\rho_4}{\searrow} & & & \\ & & V_2 & \xrightarrow{\rho_2} & F_{k_2}(E). \end{array}$$

Then consider localisation with respect to the $\mathbb{G}_m$ action on $\acute{\mathcal{M}}^{\mathrm{ss}}_{\alpha}(\acute{\tau})$ by scaling ρ_4.

□

References

[Joy21] Dominic Joyce. *Enumerative Invariants and Wall-Crossing Formulae in Abelian Categories.* Nov. 8, 2021. preprint.

[KL13] Young-Hoon Kiem and Jun Li. "A Wall Crossing Formula of Donaldson-Thomas Invariants without Chern-Simons Functional". In: *Asian Journal of Mathematics* 17.1 (Mar. 2013), pp. 63–94.