Joyce's Vertex Algebras and Moduli Spaces

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March 2023

1 Introduction

The goal of my talk is to explain the following theorem of Joyce:

Theorem 1.1. *Let T be a dg-category endowed with a Calabi-Yau orientation. Then the cohomology of M_T - the derived moduli stack parametrizing objects in T, admits a structure of a graded vertex algebra.*

We will spend most of the talk introducing the terms used in the theorem.

2 Moduli of objects

Notation 2.1. We fix a field k of characteristic zero.

Construction 2.2. Let T be a k-linear abelian category. Then we can look at the functor of points

$$m_T : CAlg_k^{classic} \to Groupoids$$

sending

$$A \mapsto (Fun_{k-lin}(T^{op}, Proj(A)^{ft}))^{\simeq}$$

the groupoid of projective A-modules of finite type.

This assignment is functorial in A: if $A \to A'$ is a map of commutative algebras, the base change functor

$$- \otimes_A A' : Proj(A)^{ft} \to Proj(A')^{ft}$$

induces a composition map

$$m_T(A) \to m_T(A')$$

Example 2.3. Let $T = [0]$ be the k-linear category with a single object and k as endomorphisms. Then $m_{[0]}$ sends

$$A \mapsto Fun_{k-lin}([0], Proj(A)^{ft})^{\simeq} \simeq (Proj(A)^{ft})^{\simeq}$$

but this is precisely the functor of points of the stack of vector bundles, so as stacks

$$m_{[0]} = Vect$$

Example 2.4. Assume $T = Mod_B^{cl}$ is the 1-category of modules over a classical associative algebra B which is finitely presented over k. Then we can show that

$$m_T(A) := \{B \otimes_k A - \text{modules which are projective and of finite type over } A\}$$

Construction 2.5. Assume again that T is Mod_B^{cl} with B associative k-algebra finitely presented over k. Then we have canonical forgetful functor

$$m_T \to Vect$$

defined on the functor of points by sending $M \in m_T(A)$ to $M \in Vect(A) = Proj(A)^{ft}$, forgeting the B-module structure. This is well-defined because of the assumption that B is finitely presented.

Proposition 2.6. *Under the same hypothesis as in the construction above, m_T is an Artin stack locally of finite type.*

Proof. Indeed, under the assumption we have the forgetful functor

$$m_T \to Vect$$

But

$$Vect = \coprod_n BGL_n$$

is an Artin stack locally of finite presentation, so to deduce the same for m_T it will be enough to show that the forgetful functor $m_T \to Vect$ is representable by affine schemes. But indeed, the fiber of $E \in Vect$ along the forgetful functor is just the affine scheme of B-module structures on E, which is of finite presentation. □

Remark 2.7. A priori, the k-points of m_T do not have to agree with the objects of T. Indeed, we have

$$m_T(k) = Fun_k(T^{op}, Proj(k)^{ft})$$

However, when T is *dualizable* with dual T^{op}, we can swap T and T^{op} to get

$$m_T(k) \simeq Fun_k(Proj(k)^{ft}, T)^{\simeq} = Obj(T)^{\simeq}$$

All this discussion was a preparation for the derived moduli of objects in a dg-category:

Notation 2.8. We denote by

- Mod_k the infinity category of k-modules.
- $dgCat_k$ the ∞-category of dg-categories
- $dgCat_k^{idem}$ the full-subcategory of idempotent complete dg-categories.

Construction 2.9. Let T be a dg-category over k. We define a functor of points evaluated on simplicial commutative k-algebras:

$$M_T : SCR_k \to SSets$$

sending

$$A \mapsto M_T(A) := Fun_{dg}(T^{op}, Perf(A))^{\simeq} = Map_{dgCat_k}(T^{op}, Perf(A))$$

the maximal ∞-groupoid of dg-functors.

Remark 2.10. The ∞-category SCR_k is what you get when you complete $Poly_k$ the category of polynomial algebras, under relative derived tensor products.

Remark 2.11. Notice that since $Perf(A)$ is an idempontent complete dg-category (Karoubi-complete) then

$$Map_{dgCat_k}(T^{op}, Perf(A)) \simeq Map_{dgCat_k^{idem}}(T^{op}, Perf(A))$$

Example 2.12. When $T = [0]$,

$$M_{[0]} = Perf$$

the derived stack of perfect complexes.

Definition 2.13. A dg-category T is said to be of finite type if there exists a dg-algebra B which is homotopically finitely presented and $\widehat{T} \simeq dg - mod_B$

Theorem 2.14 (Toen-Vaquié)**.** *Assume T is a dg-category of finite type (ex: T dualizable in $dgCat_k^{idem}$, aka smooth and proper). Then M_T is derived stack obtain as a union of open n-geometric sub-stacks which are locally of finite presentation.*

Proof. Exactly as in the context above for classical k-linear categories, the assumption that T is of finite type (I avoided the definition), there exists a well-defined forgetful map

$$M_T \to Perf$$

To show that M_T is geometric, we start by explaning that $Perf$ is a union of geometric stacks, written as

$$Perf = \bigcup Perf^{[a,b]}$$

for $Perf^{[a,b]}$ the substack of perfect complexes in tor-amplitude concentrated in degrees $[a, b]$. We show this is geometric by induction starting from the fact that $Perf^{[a,a]} = Vect$. □

Remark 2.15. Assume T is of finite type. Since M_T is locally of finite presentation by the theorem, it admits a perfect cotangent complex. Let $E \in T$ seen as a k-point in M_T. Then the tangent complex at E is given by

$$\mathbb{T}_E M_T \simeq T(E, E)[1].$$

and therefore the cotangent complex is given by

$$\mathbb{L}_E M_T \simeq T(E, E)^{\vee}[-1].$$

To compute this, one looks instead at the deformation theory of the loops as E, $\Omega_E M_T = Aut_T(E)$ classifying automorphisms of E. We then show that the tangent at the identity of E is $T(E, E)$ - all endomorphisms.

Example 2.16. When X is a smooth and proper scheme, the dg-category of perfect complexes on X, $Perf(X)$ is dualizable and we have

$$M_{Perf(X)} = \underline{Perf}(X)$$

the stack of perfect complex on X. In general, these two stacks are different and $M_{Perf(X)}$ has no clear relation to perfect complexes on X.

3 Calabi-Yau Categories

We now explain how the assumption of an orientation data on the dg-category T produces a symplectic structure on M_T that will be relevant for the construction of the vertex algebra.

For that purpose we will need to revisit the construction of Hochschild homology for dg-categories:

Definition 3.1 (Hochschild Homology of a dg-category)**.** Let T be a dg-category (or if you prefer, a small stable k-linear ∞-category). Then the k-linear yoneda-embedding

$$h_T : T \to Fun_k(T^{op}, Mod_k)$$

can be seen as an object $F_T \in Fun_k(T \otimes T^{op}, Mod_k) \simeq Fun_k^{cont}(\widehat{T \otimes T^{op}}, Mod_k)$ (the last equivalence is the k-linear yonda lemma). But in $\widehat{T \otimes T^{op}}$ we have a canonical object, namely, the bifunctor $Hom_T(-,-)$, which is a $T \otimes T^{op}$ bimodule. We define

$$HH_\bullet(T) := F_T(Hom_T(-,-)) \in Mod_k$$

Notice that

$$\widehat{T \otimes T^{op}} \simeq \widehat{T} \otimes \widehat{T^{op}}$$

in presentable dg-categories. Moreover, Hom_T is given by a continuous k-linear functor

$$Mod_k \to \widehat{T} \otimes \widehat{T^{op}}$$

that corresponds to coevaluation map exhibiting $\widehat{T^{op}}$ as a dual to $\widehat{T}$ in Pr_k^L. The composition in Pr_k^L

$$Mod_k \to \widehat{T} \otimes \widehat{T^{op}} \to Mod_k$$

is, by, definition $HH_\bullet(T)$, therefore exhibiting $HH_\bullet(T)$ as the trace of the identity of $\widehat{T}$ in Pr_k^L Explicitly, it is given by a colimit Kan extension formula

$$HH_\bullet(T) = \underset{(a,b)\to Hom_T}{colim} T(a,b)$$

Notice that when $T = Perf(R)$ is the dg-category of perfect modules over a dg-algebra R flat over k, we find

$$HH_\bullet(Perf(R)) \simeq R \underset{R\otimes R^{op}}{\otimes^{\mathbb{L}}} R$$

Definition 3.2 (Calabi-Yau dg-category)**.** Let T be a dg-category over k. An *orientation* of dimension d on T is the data of a map of mixed complexes[1]

$$HH_\bullet(T) \to k[-d]$$

where $k[-d]$ is endowed with the trivial mixed structure and such that for all objects $a, b \in T$ the composition

$$T(a,b) \otimes T(b,a) \to HH_\bullet(T) \to k[-d]$$

is non-degenerated, ie, induces an equivalence

$$T(a,b) \simeq T(b,a)^\vee[-d]$$

[1]Here mixed structure means the circle action

Remark 3.3. This form of orientation data is also called a right Calabi-Yau structure in Brav-Dyckerhoff I and Brav-Dyckerhoff II. Notice that the map of mixed complexes $HH_\bullet(T) \to k[-d]$ with $k[-d]$ with the trivial mixed structure, factors through co-invariants

$$HH_\bullet(T)_{hS^1} \to k[-d]$$

In Brav-Dyckerhoff I and Brav-Dyckerhoff II, the authors introduce a notion of left Calabi-Yau structure which seems to exist in more general situations. When T is a smooth and proper dg-category, the two notions coincide ?

Theorem 3.4 (Toën)**.** *Let T be a dg-category with a d-dimensional orientation. Then the moduli of objects M_T is (2-d)-shifted symplectic.*

Proof. Sketch: By construction, if $E \in T$, the d-Calabi-Yau structure on T produces an equivalence

$$T(E,E) \simeq T(E,E)^\vee[-d]$$

Using the formulas for the cotangent and tangent complexes computed above, we find

$$\mathbb{T}_E M_T \simeq \mathbb{L}_E M_T[2-d]$$

□

In the following we assume that T is $d = 2n$ Calabi-Yau.

Construction 3.5. Let $Exact_T$ denote the stack classifying exact sequences in T

$$[E_1 \to E_2 \to E_3]$$

Since T is stable, the map $E_1 \to E_2$ determines in a unique way the map $E_2 \to E_3$ by passing to cofibers. Therefore, we can identify the stack $Exact_T$ with $M_{T^{\Delta^1}}$ the moduli of objects in T^{Δ^1}, ie, the dg-category of morphims in T:

$$Exact_T \simeq M_{T^{\Delta^1}}$$

We consider the three maps $e_1, e_2, e_3 : Exact_T \to M_T$, sending respectively

$$e_i[E_1 \to E_2 \to E_3] = E_i$$

The projection $e_1 \times e_3 : Exact_T \to M_T \times M_T$ admits a canonical section $s : M_T \times M_T \to Exact_T$ sending

$$(E,F) \mapsto [E \to E \oplus F \to F]$$

Remark 3.6. It follows from the equivalence $Exact_T \simeq M_{T^{\Delta^1}}$ that the tangent complex at $E_1 \to_f E_2 \to E_3$ in $Exact_T$ is

$$T_{E_1 \to_f E_2 \to E_3} Exact_T \simeq T_{E_1 \to_f E_2} M_{T^{\Delta^1}}$$

given by the fiber product

$$\begin{array}{ccc} T_{E_1 \to_f E_2} M_{T^{\Delta^1}} & \longrightarrow & \mathbb{R}Hom(E_1,E_1)[1] \\ \downarrow & & \downarrow{\scriptstyle f\circ -} \\ \mathbb{R}Hom(E_2,E_2)[1] & \xrightarrow{-\circ f} & \mathbb{R}Hom(E_1,E_2)[1] \end{array}$$

corresponding to the fact that maps of arrows are given by commutative squares.

Construction 3.7. We denote by Ext the canonical perfect complex on $M_T \times M_T$ whose fibers at (E, F) are given by $\mathbb{R}Hom(E, F) = T(E, F)$. It can obtained as a relative cotangent complex as follows: first let us compute the tangent map for

$$(e_1 \times e_3) : Exact_T \to M_T \times M_T$$

$$\mathbb{T}_{Exact_T} \to (e_1 \times e_3)^* \mathbb{T}_{M_T \times M_T}$$

At a point $[E_1 \to_f E_2 \to E_3] \in Exact_T$ it is given by a map

$$\mathbb{R}Hom(E_1, E_1)[1] \underset{\mathbb{R}Hom(E_1,E_2)[1]}{\times} \mathbb{R}Hom(E_2, E_2)[1] \to \mathbb{R}Hom(E_1, E_1)[1] \times \mathbb{R}Hom(E_3, E_3)[1]$$

This map can be obtained from the diagram of cofiber sequences:

$$\begin{array}{ccccc}
\mathbb{R}Hom(E_1, E_1) & \longleftarrow & \mathbb{R}Hom(E_2, E_1) & \longleftarrow & \mathbb{R}Hom(E_3, E_1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}Hom(E_1, E_2) & \longleftarrow & \mathbb{R}Hom(E_2, E_2) & \longleftarrow & \mathbb{R}Hom(E_3, E_2) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}Hom(E_1, E_3) & \longleftarrow & \mathbb{R}Hom(E_2, E_3) & \longleftarrow & \mathbb{R}Hom(E_3, E_3)
\end{array}$$

where $\mathbb{T}_{Exact_T}[-1]$ fits as a pullback

$$\begin{array}{ccccc}
\mathbb{R}Hom(E_1, E_1) & \longleftarrow & \mathbb{T}_{Exact_T}[-1] & \longleftarrow & \mathbb{R}Hom(E_3, E_1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}Hom(E_1, E_2) & \longleftarrow & \mathbb{R}Hom(E_2, E_2) & \longleftarrow & \mathbb{R}Hom(E_3, E_2) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}Hom(E_1, E_3) & \longleftarrow & \mathbb{R}Hom(E_2, E_3) & \longleftarrow & \mathbb{R}Hom(E_3, E_3)
\end{array}$$

The fact that the the composition $\mathbb{T}_{Exact_T}[-1] \to \mathbb{R}Hom(E_1, E_3)$ vanishes, and the fact that the last row is exact, implies the existence of a factorization

$$\begin{array}{ccccc}
\mathbb{R}Hom(E_1, E_1) & \underset{u}{\longleftarrow} & \mathbb{T}_{Exact_T}[-1] & & \\
\downarrow & & \downarrow & \overset{v}{\searrow} & \\
\mathbb{R}Hom(E_1, E_2) & \longleftarrow & \mathbb{R}Hom(E_2, E_2) & & \searrow \\
\downarrow & & \downarrow & & \\
\mathbb{R}Hom(E_1, E_3) & \longleftarrow & \mathbb{R}Hom(E_2, E_3) & \longleftarrow & \mathbb{R}Hom(E_3, E_3)
\end{array}$$

The tangent map is then the direct sum of u and v:

$$(u, v) : \mathbb{R}Hom(E_1, E_1)[1] \underset{\mathbb{R}Hom(E_1,E_2)[1]}{\times} \mathbb{R}Hom(E_2, E_2)[1] \to \mathbb{R}Hom(E_1, E_1)[1] \times \mathbb{R}Hom(E_3, E_3)[1]$$

To compute the relative tangent complex of $(e_1 \times e_3)$ we remark that in general, the fiber of a map $f = (u, v) : M \to N \oplus P$ in a stable category is given by taking sucessive cofibers:

$$
\begin{array}{ccccccc}
fib(u,v) & \longrightarrow & fib(u) & \longrightarrow & M & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & N \oplus P & \longrightarrow & P \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & N & \longrightarrow & 0
\end{array}
$$

Using this, we compute:

$$
\begin{array}{ccccc}
0 & \longleftarrow & fib(u) = \mathbb{R}Hom(E_3, E_2) & & \\
\downarrow & & \downarrow & & \\
\mathbb{R}Hom(E_1, E_1) & \overset{}{\underset{u}{\longleftarrow}} & \mathbb{T}_{Exact_T}[-1] & \overset{v}{\dashrightarrow} & \\
\downarrow & & \downarrow & & \\
\mathbb{R}Hom(E_1, E_2) & \longleftarrow & \mathbb{R}Hom(E_2, E_2) & & \\
\downarrow & & \downarrow & & \\
\mathbb{R}Hom(E_1, E_3) & \longleftarrow & \mathbb{R}Hom(E_2, E_3) & \longleftarrow & \mathbb{R}Hom(E_3, E_3)
\end{array}
$$

Finally, using the last column fiber sequence, the fiber of vertical composition map is

$$
\begin{array}{ccc}
fib(u) & \longleftarrow & fib(u,v) = \mathbb{R}Hom(E_3, E_1) \\
\downarrow & & \downarrow \\
\mathbb{R}Hom(E_3, E_3) & \longleftarrow & 0
\end{array}
$$

In particular, we obtain a cofiber sequence

$$
\begin{array}{ccc}
\mathbb{R}Hom(E_3, E_1)[1] & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\mathbb{R}Hom(E_1, E_1)[1] \underset{\mathbb{R}Hom(E_1,E_2)[1]}{\times} \mathbb{R}Hom(E_2, E_2)[1] & \xrightarrow{(u,v)} & \mathbb{R}Hom(E_1, E_1)[1] \times \mathbb{R}Hom(E_3, E_3)
\end{array}
$$

exhibiting the relative tangent at a point as

$$
\mathbb{T}_{(e_1 \times e_3),[E_1 \to_f E_2 \to E_3]} = \mathbb{R}Hom(E_3, E_1)[1]
$$

Finally, using the section s we find

$$
\mathbb{T}_s \simeq s^* \mathbb{T}_(e_1 \times e_3)[-1] \simeq Ext
$$

(see here, page 36).

Remark 3.8. The Calabi-Yau structure on T induces an equivalence

$$
\sigma^*(Ext)^\vee[-2n] \simeq Ext
$$

where $\sigma : M_T \times M_T \to M_T \times M_T$ is the permutation of factors.

This won't be relevant for the talk, but I think it is worth noticing:

Theorem 3.9 (Calaque-Bozec-Scherotzke)**.**

$$T^*[2-n]M_T \simeq M_{T^{CY-n}}$$

where the r.h.s is the n-calabi-yau completion of T.

4 Operations on the moduli of objects

Construction 4.1. The moduli stack M_T carries a natural monoid law via the sum of objects in T:

$$\oplus : M_T \times M_T \to M_T$$

given by the composition

$$e_2 \circ s : M_T \times M_T \to Exact_T \to M_T$$

sending

$$(E, F) \mapsto E \oplus F$$

This monoid law is compatible with the perfect complex Ext, in the following sense that

$$\mathbb{R}Hom(E \oplus F, G) = \mathbb{R}Hom(E, G) \oplus \mathbb{R}Hom(F, G)$$

This corresponds to an equivalence of perfect complexes over $M_T \times M_T \times M_T$. Namely, consider the maps

$$\oplus \times p_3 : M_T \times M_T \times M_T \to M_T \times M_T$$

$$p_1 \times p_3 : M_T \times M_T \times M_T \to M_T \times M_T$$

$$p_2 \times p_3 : M_T \times M_T \times M_T \to M_T \times M_T$$

We have

$$(\oplus \times p_3)^* Ext \simeq (p_1 \times p_3)^* Ext \quad \oplus \quad (p_2 \times p_3)^* Ext$$

Construction 4.2. For a dg-category T and an object $E \in T$, the stabilizers of E in M_T are given by the automorphisms groups $Aut_T(E)$. These admit an action of $\mathbb{G}_m$ under

$$(\lambda, \phi) \mapsto \lambda.Id_E \circ \phi$$

.
This provides an action of the stack $B\mathbb{G}_m$ on M_T by scaling automorphisms. This action is compatible with direct sums.

Definition 4.3. We ask that the $B\mathbb{G}_m$ action is compatible with the perfect complex Ext, in the sense that

$$\begin{array}{ccc} B\mathbb{G}_m \times M_T \times M_T & \xrightarrow{(k, Ext)} & Perf \times Perf \\ \downarrow{\scriptstyle (action \times p_3)} & & \downarrow{\scriptstyle \otimes} \\ M_T \times M_T & \xrightarrow{Ext} & Perf \end{array}$$

where $k : B\mathbb{G}_m \to Perf$ classifies k as a trivial representation of $\mathbb{G}_m$.

All this is meant to ensure that $\lambda \in \mathbb{G}_m$ acts by multiplication by λ on $Ext^i(E, F)$, ie,

$$k \otimes \mathbb{R}Hom(E, F) \simeq \mathbb{R}Hom(E, F)$$

is $\mathbb{G}_m$-equivariant, where on the lhs, $\mathbb{G}_m$ acts only on k and on the r.h.s $\mathbb{G}_m$ acts via the action defined above.

Remark 4.4. The stack $B\mathbb{G}_m$ is a group stack under the tensor product of line bundles

$$\otimes : B\mathbb{G}_m \times B\mathbb{G}_m \to B\mathbb{G}_m$$

Notice that M_T is a $B\mathbb{G}_m$-module. This follows from $(\lambda_1.\lambda_2).\phi = \lambda_1.(\lambda_2.\phi)$ at the level of morphisms in T (ie, a consequence of the k-linear structure on T).

Construction 4.5. The Euler form, defined by assignment $(E, F) \in obj(T \times T) \to \mathbb{Z}$ sending

$$(E, F) \mapsto \sum_{n \geq 0} (-1)^n \, dim_k \left(Ext^i(E, F)\right)$$

is well-defined on the Grothendieck group of T

$$\chi : K_0(T) \times K_0(T) \to \mathbb{Z}$$

ie, if $E_1 \to E_2 \to E_3$ is a cofiber sequence in T, then we have

$$\chi(E_2, F) = \chi(E_1, F) + \chi(E_3, F)$$

and the same for F. This follows from the same result for a cofiber sequence of perfect complexes because in this case the euler characteristic is the trace of the identity.

Let

$$ker\chi := \{E \in K_0(T) : \chi(E, -) = 0\}$$

Then χ factors through the numerical Grothendieck group

$$K_0^{num}(T) = K_0(T)/Ker\chi$$

defining a non-degenerated pairing

$$\chi : K_0^{num}(T) \times K_0^{num}(T) \to \mathbb{Z}$$

Proposition 4.6. *The moduli of objects M_T decomposes as a disjoint union of open and closed substacks*

$$M_T = \coprod_{\alpha \in K_0^{num}(T)} M_T^{\alpha}$$

Moreover, the restriction of the perfect complex Ext to $M_T^{\alpha} \times M_T^{\beta}$ has precisely rank $\chi(\alpha, \beta)$.

5 Vertex algebra structure on the homology of the M_T

We can finally start explaining the main theorem of this talk. Summarizing what we have so far, under the assumption that T is a $d = 2n$ CY-dg-category, the derived stack M_T carries an action of $B\mathbb{G}_m$ compatible with the perfect complex $Ext \in Perf(M_T \times M_T)$ and with the CY-structure.

Assumption 5.1. In what follows, we will take the Betti homology of stacks. This can be obtained by first taking the topological realization and then computing Betti homology. We will not give the details here and simply assume we have a nice enough homology theory for stacks. See A. Blanc thesis.

Construction 5.2. The action of $B\mathbb{G}_m$ on M_T provides an action on homology

$$H_*(B\mathbb{G}_m) \otimes H_*(M_T) \to H_*(B\mathbb{G}_m \times M_T) \to H_*(M_T)$$

The first step to understand this action is to compute the homology of $B\mathbb{G}_m$.

Remark 5.3. The topological realization of $B\mathbb{G}_m$ is $\mathbb{C}P^\infty$ the classifying space of complex line bundles. To see this, one observes that the topological realization of $\mathbb{G}_m$ is given by its complex points, ie, $\mathbb{C}^*$ which as an H-space is homotopy equivalent to $S^1 = B\mathbb{Z}$. Since the topological realization commutes with colimits, in particular, it commutes with B and therefore the topological realization of $B\mathbb{G}_m$ is $B^2\mathbb{Z} = K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$.

Its cohomology $H^*(\mathbb{C}P^\infty, \mathbb{Q})$ is the formal power series algebra with a generator in degree 2, c_1, corresponding to the first chern class of the universal line bundle.

$$H^*(\mathbb{C}P^\infty, \mathbb{Q}) \simeq \mathbb{Q}[[c_1]]$$

Now, both $B\mathbb{G}_m$ and $\mathbb{C}P^\infty$ have natural group laws given by the tensor product of line bundles. In particular, the cohomology $H^*(\mathbb{C}P^\infty, \mathbb{Q}) \simeq \mathbb{Q}[[c_1]]$ inherits the structure of an Hopf algebra with comultiplication given by $u \mapsto u \otimes 1 + 1 \otimes 1$ (ie, the additive group law). It follows that the homology $H_*(\mathbb{C}P^\infty, \mathbb{Q})$ acquires a multiplication law dual to the Hopf structure. A general result for Hopf algebras (see Hatcher Chapter 3C.11) characterizes $H_*(\mathbb{C}P^\infty, \mathbb{Q})$ as the algebra of divided power series in one variable $t \in H_2(\mathbb{C}P^\infty, \mathbb{Q})$ (dual to c_1) , ie

$$t^n * t^m = \binom{n+m}{n} t^{n+m} = \frac{(n+m)!}{n!.m!} t^{n+m}$$

To prove this, as in Hatcher's book, we use the fact that the comultiplication Δ in $H^*(\mathbb{C}P^\infty, \mathbb{Q}) \simeq \mathbb{Q}[[c_1]]$ satisfies

$$\Delta(c_1^n) = (c_1 \otimes 1 + 1 \otimes c_1)^n$$

and check that the contribution for $t^i * t^{n-i}$ comes from the product

$$\binom{n}{i} c_1^i \otimes c_1^{n-i}$$

Notation 5.4. We denote by D the action of t on $H_*(M_T)$

$$t : H_*(M_T) \to H_*(M_T)$$

And by D^n, the divided powers

$$t^n : H_*(M_T) \to H_*(M_T)$$

Remark 5.5. Since we have the $\mathbb{Q}$-linear dual

$$Hom_{\mathbb{Q}}(H_*(B\mathbb{G}_m),\mathbb{Q}) = \mathbb{Q}[[z]]$$

with $z = c_1$ as above. In particular, the action of $B\mathbb{G}_m$ on M_T, gives a map

$$H_*(M_T) \to H_*(M_T)[[z]]$$

Theorem 5.6 (Joyce)**.** *The Betti homology*

$$H_*(M_T) = \bigoplus_{\alpha \in K_0^{num}(T)} H_*(M_T^\alpha)$$

admits a structure of a $K_0^{num}(T)$-graded vertex algebra with:

- *D given by the action of $t \in H_2(B\mathbb{G}_m)$;*
- *$|0>= Im(0 : H_*(Spec(k) \to H_*(M_T)$;*
- *Given $u \in H_*(M_T^\alpha)$, $v \in H_*(M_T^\beta)$,*

$$Y(u,z)v := (-1)^{\chi(\alpha,\beta)} \sum_{i,j\geq 0} z^{\chi(\alpha,\beta)+\chi(\beta,\alpha)-i+j} (\bigoplus_{*} (action \times Id)_*)[$$

$$\underbrace{\underbrace{t^j}_{H_{2j}(B\mathbb{G}_m)} \boxtimes (u \boxtimes v) \cap c_i(Ext_{|_{M_T^\alpha \times M_T^\beta}} \oplus \sigma^* Ext_{|_{M_T^\beta \times M_T^\alpha}})}_{H_*(B\mathbb{G}_m \times M_T \times M_T)}]$$

Proof. As far as i understood, the proof follows by diagram chasing, using the diagrams given in this talk for the action of $B\mathbb{G}_m$ on M_T and the compatibility with Ext and the CY-structure. □

To conclude the talk, let me mention another result proved by Joyce. Recall that in the previous lectures we proved that if V is a vertex algebra wth translator operator D, then V/D is a Lie algebra. In our current example, $V = H_*(M_T)$ and D is given by action of the element $t \in H_2(B\mathbb{G}_m)$.

Construction 5.7. Let us denote by

$$M_T^{pl} := M_T/B\mathbb{G}_m$$

the derived stacky quotient. Since $B\mathbb{G}_m$ only acts on morphisms, the k-points of $M_T/B\mathbb{G}_m$ and M_T are the same. The difference is on automorphisms where we mod out by the scaling:

$$Aut_{M_T^{pl}}(E) = Aut_T(E)/\phi \sim \lambda\phi$$

Theorem 5.8 (Joyce)**.** *The map $H_*(M_T) \to H_*(M_T^{pl})$ is isomorphic to the quotient map*

$$H_*(M_T) \to H_*(M_T)/D$$

In particular, it exhibits $H_*(M_T^{pl})$ as graded a Lie algebra.

Example 5.9. When X is a smooth projective scheme we have $M_{Perf(X)} = \underline{Perf}(X)$ the derived stack of perfect complexes on X. When X is 2n-calabi-yau we get a vertex algebra structure on $H_*(\underline{Perf}(X)$. When X is $(2n+1)$-calabi-yau, we get $c_i(Ext) = 0$ and we get an abelian vertex algebra.