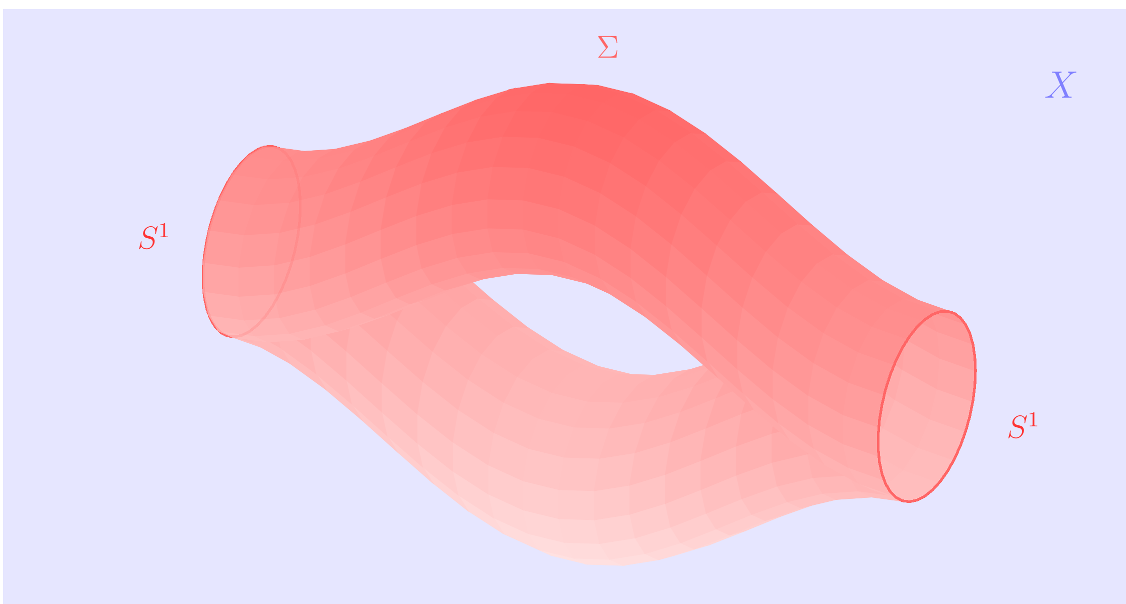
AFFINE W ALGEBRAS: THE ABSOLUTE BASICS

ALEXEI LATYNTSEV

1.1. **Chiral quantisation.** What is a vertex algebra? Something like a two dimensional conformal field theory.

And what is the simplest example of a $2d$ CFT? The *sigma model* with target space X (a scheme, manifold, etc.), which loosely speaking is the functor $\mathrm{Maps}(-, X)$.



So it takes the circle to the free loop space $LX = \mathrm{Maps}(S^1, X)$, and for Σ a surface with two boundary circles we get a correspondence

$$\begin{array}{ccc} & \mathrm{Maps}(\Sigma, X) & \\ \swarrow & & \searrow \\ LX & & LX \end{array}$$

In algebraic geometry, there are many existing ways to formalise LX, but none work for these purposes. Instead restrict to the component $0 \in \pi_1 X = \pi_0 LX$ corresponding to contractible loops in X, the *jet space* $J_\infty X = \mathrm{Maps}(D, X)$. Here $D = \mathrm{Spec}k[[t]]$ is the formal disk.

Proposition 1.2. *If X is any scheme, $J_\infty X$ is a (ind) scheme and $\mathcal{O}(J_\infty X)$ is a holomorphic vertex algebra.*

Proof. The vector field ∂_t on D induces one on the jet space, hence equips its function ring with a derivation. □

For instance,

$$\mathcal{O}(J_\infty \mathbf{A}^1) \;\simeq\; k[x_{-1}, x_{-2}, \dots], \qquad \partial x_{-n} \;=\; -n x_{-n-1}.$$

Every vertex algebra is canonically filtered by

$$V^{\leq n} \;=\; (\text{span of } \alpha_{-n_1-1}\beta_{-n_2-1}\cdots|0\rangle \;:\; n_1+n_2+\cdots \;\leq\; n)$$

and the associated graded $\mathrm{gr}V$ is a (Poisson) holomorphic vertex algebra.

Definition 1.3. V is a *chiral quantisation* of X if $\mathrm{gr}V \simeq \mathcal{O}(J_\infty X)$.

This implies that X must have a Poisson structure.

1.4. **Examples.** We now have two orthogonal notions of quantisation: *chiral quantisation* which introduces S^1's and *filtered quantisation* which makes things noncommutative, e.g.

$$\begin{array}{ccc} V^k(\mathfrak{g}) & \xrightarrow{\ \mathrm{gr}\ } & \mathcal{O}(J_\infty \mathfrak{g}^*) \\ \text{Zhu algebra}\downarrow & & \downarrow\text{Zhu algebra} \\ U(\mathfrak{g}) & \xrightarrow{\ \mathrm{gr}\ } & \mathcal{O}(\mathfrak{g}^*) \end{array}$$

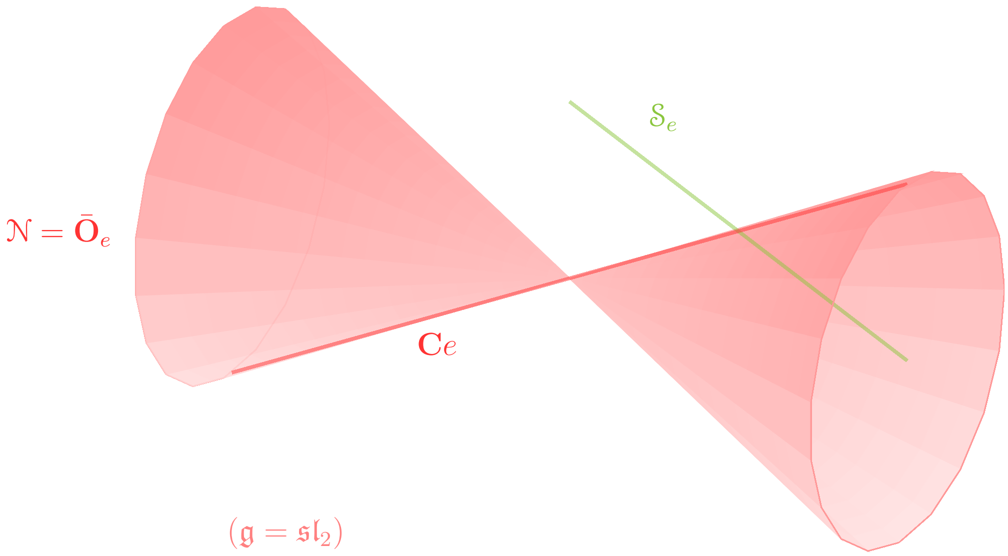
Note that $\mathfrak{g}^*$ is Poisson. To get more interesting examples, take the orbit

$$\mathbf{O}_e \;=\; G\cdot e \;\subseteq\; \mathfrak{g} \;\simeq\; \mathfrak{g}^*$$

of a nilpotent element $e \in \mathfrak{g}$ of a finite dimensional semisimple Lie algebra over $\mathbf{C}$, e.g.

$$\mathfrak{sl}_2 \;=\; \mathbf{C}\left\{e = \left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right),\; h = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right),\; f = \left(\begin{smallmatrix} 0 & \\ 1 & 0 \end{smallmatrix}\right)\right\}.$$

The transverse slice $\mathcal{S}_e = e + \ker[f, -]$ also has a Poisson structure, as do the intersections $\mathcal{S}_{e'} \cap \overline{\mathbf{O}}_e$ for pairs of nilpotent e, e'.



The *W algebra* $W(\mathfrak{g}, e)$ and *affine W algebra* $\mathcal{W}^k(\mathfrak{g}, e)$ are (vertex) algebras fitting into the diagram

$$\begin{array}{ccc} \mathcal{W}^k(\mathfrak{g}, e) & \overset{\text{gr}}{\rightsquigarrow} & \mathcal{O}(J_\infty \mathcal{S}_e) \\ \text{Zhu algebra} \downarrow & & \downarrow \text{Zhu algebra} \\ W(\mathfrak{g}, e) & \overset{\text{gr}}{\rightsquigarrow} & \mathcal{O}(\mathcal{S}_e) \end{array}$$

Its function ring being an associated graded, $\mathcal{S}_e$ inherits a $\mathbf{G}_m$ action. Which? The *Kazhdan scaling* action on $\mathfrak{g}^*$, which in the $\mathfrak{sl}_2$ case is

$$t \cdot e \;=\; e, \qquad t \cdot h \;=\; t^2 h, \qquad t \cdot f \;=\; t^4 f.$$

Moreover there is a unipotent subgroup $N \subseteq G$, being $\{(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix})\}$ in the $\mathfrak{sl}_2$ case, with

Proposition 1.5. *$\mathcal{S}_e = \mu^{-1}(e)/N$ is the Hamiltonian reduction of the action of N on $\mathfrak{g}^*$.*

This immediately suggests a form for the finite W algebra:

$$\mathcal{O}(\mathcal{S}_e) \;=\; \mathcal{O}(e + (\mathfrak{g}/\mathfrak{n})^*)^N, \qquad\qquad W(\mathfrak{g}, e) \;=\; (U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbf{C}_e)^N$$

where $\mathbf{C}_e$ is the one dimensional representation of $\mathfrak{n}$ induced by the character $e \in \mathfrak{g} \simeq \mathfrak{g}^*$.

Proposition 1.6. (Kostant) *If e is principal then*

$$\mathbf{C}[\mathfrak{t}^*]^W \;=\; Z(U(\mathfrak{g})) \;\xrightarrow{\sim}\; W(\mathfrak{g},e)$$

is a polynomial algebra, e.g. this is the case for $\mathfrak{g}=\mathfrak{sl}_2$ and nonzero e.

1.7. **BRST reduction.** To affinise this, unwrap both steps of the above construction:

1) *Subspace.* To understand a closed subscheme $Z \subseteq X$, we can use the ideal sheaf $\mathcal{I} = \ker(\mathcal{O}_X \to \mathcal{O}_Z)$ to give a complex

$$\cdots \;\to\; \wedge^2\mathcal{I} \;\to\; \mathcal{I} \;\to\; \mathcal{O}_X \qquad \rightsquigarrow \qquad \cdots \;\to\; \wedge^2\mathcal{I}/\mathcal{I}^2 \;\to\; \mathcal{I}/\mathcal{I}^2 \;\to\; \mathcal{O}_Z.$$

When the closed embedding is nice (lci), the latter is exact, so we have expressed $\mathcal{O}_Z$ in terms of the conormal cone $\mathcal{I}/\mathcal{I}^2$.

2) *Quotient.* If G acts on X, then the quotient stack X/G has a simplicial resolution

$$\cdots \;\substack{\to\\\to\\\to\\\to}\; G^2\times X \;\substack{\to\\\to\\\to}\; G\times X \rightrightarrows \; X \;\to\; X/G$$

Thus by taking alternating sums of pullbacks, we get a complex

$$\cdots \;\leftarrow\; \mathcal{O}(G)^{\otimes 2}\otimes \mathcal{O}(X) \;\leftarrow\; \mathcal{O}(G)\otimes\mathcal{O}(X) \;\leftarrow\; \mathcal{O}(X) \;\leftarrow\; \mathcal{O}(X)^G$$

Similarly,[1] the action of $\mathfrak{g}$ by vector fields gives a resolution

$$\cdots \;\leftarrow\; \wedge^2\mathfrak{g}^*\otimes\mathcal{O}(X) \;\leftarrow\; \mathfrak{g}^*\otimes\mathcal{O}(X) \;\leftarrow\; \mathcal{O}(X) \;\leftarrow\; \mathcal{O}(X)^G.$$

Thus if we consider $\mathcal{S}_e \simeq (e+(\mathfrak{g}/\mathfrak{n})^*)/N$, since the conormal bundle of $e+(\mathfrak{g}/\mathfrak{n})^*\subseteq\mathfrak{g}^*$ is $\mathfrak{n}$, we can realise $\mathcal{O}(\mathcal{S}_e)$ as the zeroeth cohomology of the total complex of

$$\begin{array}{ccccc}
 & & & & \vdots \\
 & & & & \uparrow \\
 & & \cdots & \longrightarrow & \mathcal{O}(\mathfrak{g}^*)\otimes\wedge^2\mathfrak{n}^* \\
 & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & \mathfrak{n}\otimes\mathcal{O}(\mathfrak{g}^*)\otimes\mathfrak{n}^* & \longrightarrow & \mathcal{O}(\mathfrak{g}^*)\otimes\mathfrak{n}^* \\
\uparrow & & \uparrow & & \uparrow \\
\cdots \to \wedge^2\mathfrak{n}\otimes\mathcal{O}(\mathfrak{g}^*) & \longrightarrow & \mathfrak{n}\otimes\mathcal{O}(\mathfrak{g}^*) & \longrightarrow & \mathcal{O}(\mathfrak{g}^*)
\end{array}$$

and $W(\mathfrak{g},e)$ as the zeroeth cohomology of

[1] I do not actually know how to derive the below using something like the above.

$$\begin{array}{ccccccc}
 & & & & & & \vdots \\
 & & & & & & \uparrow \\
 & & & & \cdots & \longrightarrow & U(\mathfrak{g}) \otimes \wedge^2\mathfrak{n}^* \\
 & & & & \uparrow & & \uparrow \\
 & & \cdots & \longrightarrow & \mathfrak{n} \otimes U(\mathfrak{g}) \otimes \mathfrak{n}^* & \longrightarrow & U(\mathfrak{g}) \otimes \mathfrak{n}^* \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & \wedge^2\mathfrak{n} \otimes U(\mathfrak{g}) & \longrightarrow & \mathfrak{n} \otimes U(\mathfrak{g}) & \longrightarrow & U(\mathfrak{g})
\end{array}$$

Note that this double complex $\wedge^\bullet\mathfrak{n} \otimes U(\mathfrak{g}) \otimes \wedge^\circ\mathfrak{n}^*$ is a graded algebra, and its differential is defined to be

$$d \;=\; [\chi, -], \qquad \chi \;=\; x^i(\underline{x}_i - e(x^i)) - \frac{1}{2}x^i x^j [x_i, x_j]$$

where x_i and x^i vary over dual bases of $\mathfrak{n}$ and $\mathfrak{n}^*$, and $\underline{x}_i$ is viewed as an element of $\mathfrak{g}$.

1.8. **Affine W algebras.** The more correct way of viewing the above is $\wedge^\bullet\mathfrak{n} = U(\mathfrak{n}[1])$ for the abelian dg Lie algebra $\mathfrak{n}[1]$.

In the affine case, you replace

$$U(\mathfrak{n}[1] \oplus \mathfrak{n}^*[-1]) \otimes U(\mathfrak{g}) \qquad \rightsquigarrow \qquad V^1(\mathfrak{n}[1] \oplus \mathfrak{n}^*[-1]) \otimes V^k(\mathfrak{g})$$

where $\mathfrak{n}[1] \oplus \mathfrak{n}^*[-1]$ is given the obvious pairing[2] and

$$[\chi, -] \qquad \rightsquigarrow \qquad \int x^i(z)(x_i(z) - e(x^i)) - \frac{1}{2} : x^i(z)x^j(z)[x_i, x_j](z) : \; dz$$

then a vertex algebra structure is inherited from the double complex just as before:

Theorem 1.9. *The zeroeth cohomology of the double complex is a vertex algebra, called the* affine W algebra $\mathcal{W}^k(\mathfrak{g}, e)$.[3]

This process is called *semiinfinite cohomology.*[4] For instance,

1) When $\mathfrak{g} = \mathfrak{gl}_1$ and $e = 0$, this is just the Heisenberg vertex algebra $V^k(\mathfrak{gl}_1)$.

[2] which is needed to define $\mathfrak{n}[1]\widehat{\oplus\, \mathfrak{n}^*}[-1]$ and hence its induced representation $V^1(\mathfrak{n}[1] \oplus \mathfrak{n}^*[-1])$.

[3] If e is principal, all other cohomologies are zero. I do not know if this is true in general.

[4] This is because the horizontal and vertical resolutions compute Lie algebra homology and cohomology for $\mathfrak{n}$ acting on $U(\mathfrak{g})$.

2) The simplest interesting case is the *Virasoso* vertex algebra $\mathcal{W}^k(\mathfrak{sl}_2, e)$, which is generated by a single field $L(z)$ subject to

$$L(z)L(w) \;\sim\; \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w}$$

where the number

$$c \;=\; 1 - \frac{6(k+1)^2}{k+2}$$

is called the *central charge*. Its Lie algebra of modes L_n satisfies the *Virasoro relations*

$$[L_n, L_m] \;=\; (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{n+m,0}$$

so it is a central extension of Lie algebra of vector fields $z^{n+1}\partial$ on the punctured formal disk.

3) When

$$\mathfrak{g} \;=\; \mathfrak{sl}_n, \qquad e \;=\; \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

one can show that $\mathcal{W}^k(\mathfrak{sl}_n, e)$ is generated by elements $W_2, W_3, ..., W_n$ of degrees $2, 3, ..., n$.

1.10. **Coming up next time.** Relation to algebraic surfaces, instantons, geometric Langlands, ...

Question 1.11. *You can interpret* $U(\mathfrak{g})_0$ *and* $W(\mathfrak{g}, e)_0$ *geometrically,*[5] *in terms of (micro)differential operators on* G/B *and its Hamiltonian reduction. Similarly, there is a geometric description of* $V^k(\mathfrak{g})$ *as delta functions on the affine Grassmannian. Fill in the blank:*

$$\begin{array}{ccc} V^k(\mathfrak{g}) & \xrightarrow{\text{Ham red.}} & \mathcal{W}^k(\mathfrak{g}, e) \\ {\scriptstyle\text{Zhu algebra}}\downarrow & & \downarrow \\ U(\mathfrak{g}) & \xrightarrow{\text{Ham red.}} & W(\mathfrak{g}, e) \end{array} \qquad\qquad \begin{array}{ccc} \delta_{\mathrm{Gr}_G} & \xrightarrow{\text{Ham red.}} & ??? \\ \downarrow & & \downarrow{\scriptstyle\text{Zhu algebra}} \\ \mathcal{E}_{G/B} & \xrightarrow{\text{Ham red.}} & \overline{\mathcal{E}}^N_{G/B} \end{array}$$

[5]The subscript $_0$ means we have quotiented by the centre $Z(U(\mathfrak{g})) \subseteq U(\mathfrak{g})$.

What is the relation between ??? and the Virasoro uniformation description of Vir_c *in terms of* $\mathcal{M}_{g,n}$*?*

Question 1.12. $\mathcal{W}^{k_{crit}}(\mathfrak{g}, e)_0$ *is a chiral quantisation of* $\mathrm{gr}W(\mathfrak{g}, e)_0 = \mathcal{S}_e \cap \mathcal{N}$*. What are vertex algebras attached to other symplectic singularities, like* $\mathbf{C}^2/\Gamma$*,* $\mathrm{Sym}S$ *for* S *an smooth algebraic surface or* $\mathbf{C}^2/\Gamma$*, or Nakajima quiver varieties?*

2. The physics meaning of CoHAs

2.1. Warning: I am not a physicist, and everything below except the Theorems is likely to contain errors. The questions are just things that I don't know the answer to, many are probably known.

2.2. From the physics point of view, cohomological Hall algebras are an example of the (expected) functor

$$\text{BPS states} \ : \ 4d \ \mathcal{N}=2 \ \text{SCFTs} \ \to \ \text{Associative algebras}.$$

It is important to stress that the left side category (and hence the functor) has not been defined mathematically.

However, any correct eventual definition must include the following examples from physics (and hence we should have CoHAs attached to each):

1) *Gauge theories.* Attached to any reductive group with representation (G,V) (see [BFN16ii, Te21]).

1)' *Quiver gauge theories.* Attached to any ADE or affine ADE quiver with representation (Q,V) (see [BFN16i]).

2) *Theories of class S.* Attached to any simple algebraic group G and compact Riemann surface Σ, there is the conjectural $6d$ SCFT $\mathcal{T}[G]$ (see [Wi06]) and compactifying it over the Riemann surface gives
$$\mathcal{S}(G,\Sigma) \ = \ \int_\Sigma \mathcal{T}[G].$$

2)' *Yang Mills* with gauge group G and coupling constant $\tau\in\mathbf{H}$ is simply $\mathcal{S}(G,E_\tau)$ where $E_\tau=\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ is an elliptic curve.

3) *Calabi Yau threefold* compactifications of $10d$ string theory.

Kontsevich and Soibelman first considered example 3) for $X=\mathbf{C}^3$ or more generally the "toy model" for a Calabi Yau threefold: the Jacobi algebra $\mathrm{Jac}(Q,W)$ of a quiver with potential.[6]

[6] Taking Q the quiver with one dot and three loops x,y,z, and
$$W \ = \ x[y,z] \ + \ y[z,x] \ + \ z[x,y],$$

2.3. We also expect a space called the (*quantised*) *Coulomb branch* of any such theory

$$\begin{array}{ccc} & \overset{\mathcal{M}_{C,\hbar}}{\nearrow} & \text{Quantised symplectic singularity} \\ & & \downarrow \text{gr} \\ 4d\ \mathcal{N}=2\ \text{SCFTs} & \overset{\mathcal{M}_C}{\to} & \text{Symplectic singularities} \end{array}$$

Thus $\mathcal{M}_C(\mathcal{T})$ is a variety and $\mathcal{M}_{C,\hbar}(\mathcal{T})$ is a filtered algebra with associated graded $\mathcal{O}\left(\mathcal{M}_C(\mathcal{T})\right)$.

It is expected that there is a map of associative algebras

$$\text{CoHA}(\mathcal{T}) \ \to \ \mathcal{M}_{C,\hbar}(\mathcal{T})$$

from the (equivariant) CoHA. Moreover, the map should extend to the double of the CoHA, on which it is a surjection (see [So20]).

2.4. Given a $4d$ SCFT $\mathcal{T}$, we may integrate it over a circle to produce a $3d$ SCFT $\int_{S^1}\mathcal{T}$. We can likewise take its Coulomb branch, which will be a toric fibration

$$\mathcal{M}\left(\int_{S^1}\mathcal{T}\right) \ \to \ \mathcal{M}_C(\mathcal{T}),$$

and likewise this map quantises.

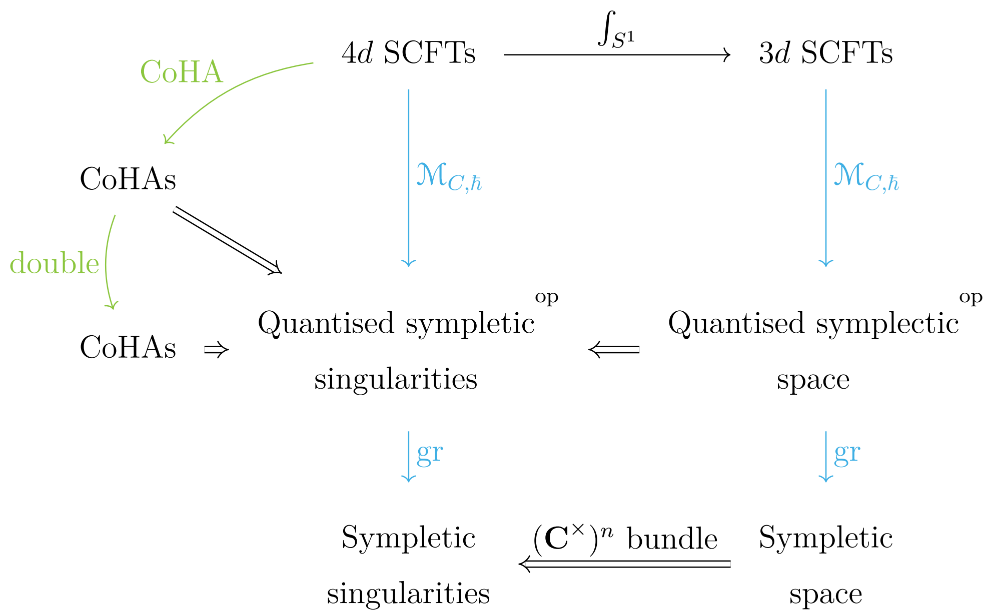
2.5. The involution $\tau \mapsto -1/\tau$ on the space $\mathcal{M}_{1,1} = \mathbf{P}(4,6)$ of elliptic curves induces "S dual" $4d$ theories, after also swapping $G \leftrightarrow G^L$:

$$\begin{array}{ccc} \mathcal{T}[G] & \longleftarrow\text{-----}\longrightarrow & \mathcal{T}[G^L] \\ \downarrow \int_E & & \downarrow \int_{\check{E}} \\ \text{super } 4d \text{ Yang Mills}[G] & \overset{\text{[KW06]}}{\underset{S \text{ duality}}{\longleftrightarrow}} & \text{super } 4d \text{ Yang Mills}[G^L] \\ \downarrow \int_{S^1} & & \downarrow \int_{S^1} \\ \text{super } 3d \text{ Yang Mills}[G] & \overset{\text{[SW96]}}{\underset{\text{Symplectic duality}}{\longleftrightarrow}} & \text{super } 3d \text{ Yang Mills}[G^L] \end{array}$$

This should extend to a $\mathbf{Z}/2$ action on the set of all $4d$ SCFTs, and on $3d$ SCFTs.

we get back $\text{Jac}(Q,W) = \mathbf{C}[x,y,z] = \mathcal{O}(\mathbf{C}^3)$.

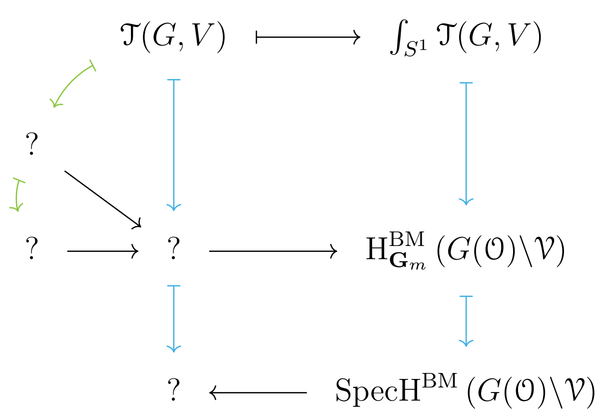
2.6. **Summary.** Expected picture:



and this diagram has a $\mathbf{Z}/2$ action.

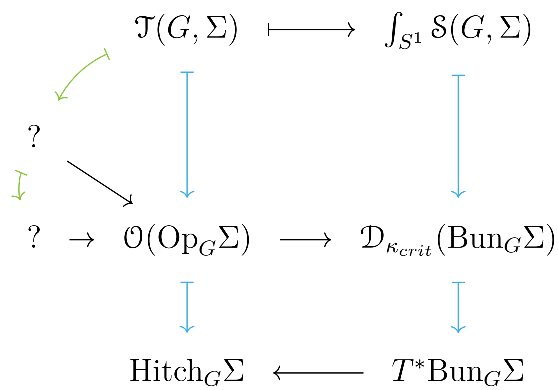
For example (many of the ?'s are probably known):

1) *Gauge theories.*



Here $\mathcal{V} = V \times_G \mathrm{Gr}_G = V \times_G G(\mathcal{K})/G(\mathcal{O})$ is the induced vector bundle over the affine Grassmannian Gr_G, and the (equivariant) Borel Moore homology is an algebra by convolution.

2) *Theories of class S.*

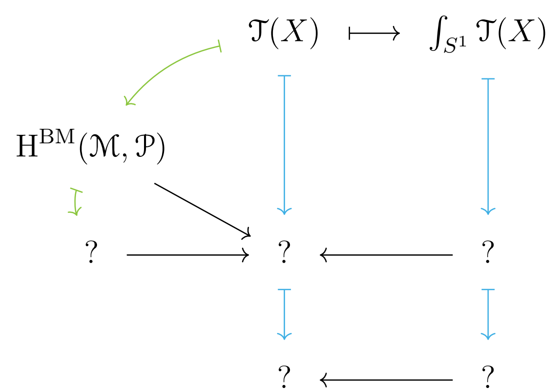


Here we have the *Hitchin base*

$$\mathrm{Hitch}_G\Sigma \;=\; \Gamma\left(\Sigma, \omega_\Sigma \otimes \mathfrak{g}^*//G\right) \;\simeq\; \bigoplus \Gamma(\Sigma, \omega_\Sigma^{d_i})$$

where d_i are the *fundamental weights* of G. Given a Higgs bundle $\varphi \in T^*\mathrm{Bun}_G\Sigma$, we pick a representation V of G, write $\mathcal{V}$ the induced vector bundle and send φ to the characteristic polynomial of $\varphi : \mathcal{V} \to \mathcal{V}$.[7] The cotangent bundle quantises to differential operators at critical level, and the Hitchin base to functions on the space of G opers on Σ (see [BD91]).

3) *Calabi Yau threefold* compactifications of $10d$ string theory. If X is a Calabi Yau threefold, consider its moduli stack $\mathcal{M} = \mathcal{M}_{\mathrm{Coh}X}$ of coherent sheaves.[8]



Here we have used that $\mathcal{M}$ is a -1 symplectic space, so by Joyce carries a perverse sheaf $\mathcal{P}$. This is especially easy when $X = K_S$ is the canonical bundle of an algebraic surface,[9] in which case $\mathcal{M}_{\mathrm{Coh}X} = T^*[-1]\mathcal{M}_{\mathrm{Coh}S}$ and $\mathcal{P}$ is a sheaf of vanishing cycles,

[7] This implies that we should probably also pick a representation of G at the start? But for some reason this is never done

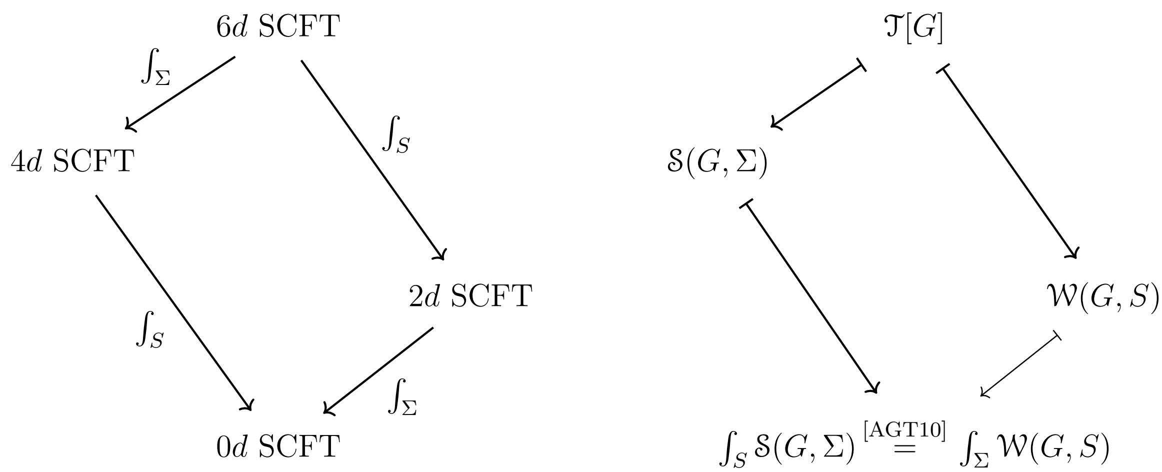
[8] Presumably we can replace $\mathrm{Coh}X$ with an arbitrary three Calabi Yau category.

[9] Or more generally, presumably any (deformed) three Calabi Yau completion of a dg category.

with

$$\mathrm{H}^{\mathrm{BM}}(\mathcal{M}_{\mathrm{Coh}S}) \;=\; \mathrm{H}^{\mathrm{BM}}(\mathcal{M}_{\mathrm{Coh}X}, \mathcal{P}).$$

2.7. **The AGT correspondence.** From now on we will focus solely on theories of class S.



Compactifying $\mathcal{T}[G]$ along an algebraic surface S gives a $2d$ SCFT $\mathcal{W}(G,S)$, which is more or less a vertex algebra $\mathcal{W}(G,S)$. The AGT conjecture says the number we get $\int_\Sigma \mathcal{W}(G,S)$ is the same as the number we get $\int_\Sigma \mathcal{W}(G,S)$. The first is (the dimension of) the space of conformal blocks(?) and the second is called the Nekrasov partition function.

Question 2.8. *How does $\mathcal{W}(G,S)$ fit into the diagrams of last section?*

2.9. In the flat case $S = \mathbf{A}^2$, these vertex algebras are expected be affine W algebras attached to regular nilpotent elements $\mathcal{W}(G,\mathbf{A}^2) \overset{?}{=} \mathcal{W}^k(\mathfrak{g}, \mathrm{reg})$, so for instance $\mathcal{W}(\mathrm{GL}_1, \mathbf{A}^1)$ should be the rank one Heisenberg vertex algebra.

More generally, the following Theorem in [Na99] suggests that $\mathcal{W}(\mathrm{GL}_1, S)$ should be a Heisenberg vertex algebra on $\mathrm{H}_\bullet(S)$:

Theorem 2.10. (Grojnowski, Nakajima) *The Borel Moore homology of the Hilbert scheme of points of a smooth quasiprojective algebraic surface S has a (Heisenberg) vertex algebra structure:*

$$\mathrm{H}^{\mathrm{BM}}_\bullet(\mathrm{Hilb}S) \;=\; V_\kappa(\mathfrak{h}), \qquad\qquad (\mathfrak{h},\kappa) \;=\; \big(\mathrm{H}^{2\bullet}(S), \textstyle\int(-)\cdot(-)\big) \;\oplus\; \big(\mathrm{H}^{2\bullet+1}(S), 0\big)\,.$$

Proof sketch. Here is how to define the vertex algebra structure on Hilb$S$. Consider *Hecke modifications* of a dimension zero subscheme $Z \in$ Hilb$S$ by a (length ℓ) dimension zero coherent sheaf:

$$\begin{array}{ccccc} & & \mathrm{Ext}_\ell S & \to & \mathrm{Ext} S \\ & \swarrow & & \swarrow & & \searrow \\ \mathrm{Hilb} S \times \mathrm{Coh}^1_{0,\ell} S & \to & \mathrm{Hilb} S \times \mathrm{Coh}_0 S & & & & \mathrm{Hilb} S \end{array}$$

If we compose with the map $\mathrm{Coh}^1_{0,\ell} S \to S$ taking support, we get the correspondence

$$\begin{array}{ccc} & \mathrm{Ext}_\ell S & \\ \swarrow & & \searrow \\ \mathrm{Hilb}^n S \times S & & \mathrm{Hilb}^{n+\ell} S \end{array} \tag{1}$$

and a second correspondence by moving S to the right side of (1), and hence[10] a pair of maps for every nonnegative integer ℓ

$$\mathfrak{p}_{-\ell} \ : \ \mathrm{H}^{\mathrm{BM}}_\bullet(\mathrm{Hilb} S) \otimes \mathrm{H}^{\mathrm{BM}}_\bullet(S) \ \to \ \mathrm{H}^{\mathrm{BM}}_\bullet(\mathrm{Hilb} S),$$
$$\mathrm{H}^{\mathrm{BM}}_\bullet(\mathrm{Hilb} S) \ \leftarrow \ \mathrm{H}^{\mathrm{BM}}_\bullet(\mathrm{Hilb} S) \otimes \mathrm{H}^{\mathrm{BM}}_\bullet(S) \ : \ \mathfrak{p}_\ell.$$

We can interpret this as $\mathfrak{p}_{\alpha,\pm\ell} \in \mathrm{End}\left(\mathrm{H}^{\mathrm{BM}}_\bullet(\mathrm{Hilb} S)\right)$ for every dual Borel Moore homology class $\alpha \in \mathrm{H}^{\mathrm{BM}}_\bullet(S)^\vee$. By Gottsche's formula, these fields generate $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathrm{Hilb} S)$. □

2.11. We need to justify why this has anything to do with $\mathcal{W}(GL_1, S)$: where does GL_1 show up? Well, the Hilbert scheme parametrises dimension zero subschemes, which is the same as a dimension zero quotient of $\mathcal{O}_S$:

$$0 \ \to \ \mathcal{I} \ \to \ \mathcal{O}_S \ \to \ \mathcal{Q} \ \to \ 0 \qquad\qquad \dim \mathrm{Supp} \mathcal{Q} \ = \ 0.$$

It follows that $\mathcal{I}$ is a *torsion free* coherent sheaf. Moreover, for any torsion free sheaf $\mathcal{E}$, its double dual $\mathcal{E}^{\vee\vee}$ is a vector bundle and the inclusion is an embedding:

$$0 \ \to \ \mathcal{E} \ \to \ \mathcal{E}^{\vee\vee} \ \to \ \mathcal{Q} \ \to \ 0.$$

[10]We have swept under the rug showing that (1) is the sort of correspondence that we can pull-push Borel Moore classes along, i.e. the left map is quasismooth and the right map is proper. We need to prove the same for the correspondence where S is moved to the right.

In particular, if $\mathcal{E}$ has generic rank one, then we can apply $\otimes(\mathcal{E}^{\vee\vee})^{-1}$ to this diagram to get an element of the Hilbert scheme, showing

$$\mathrm{Coh}^{tf}_{GL_1} S \;\simeq\; \mathrm{Hilb} S \times \mathrm{Pic} S$$

as stacks. Thus if there are no nontrivial line bundles on S then $\mathrm{Coh}^{tf}_{\mathrm{GL}_1} S$ has good moduli space $\mathrm{Hilb}S$.

Another relevant construction is attached to algebraic surfaces with divisors (S, D). If we define $\mathrm{Coh}^{tf}_{\mathrm{GL}_1}(S, D)$ to be the stack of rank one torsion free sheaves along with a trivialisation in some neighbourhood of D, then

$$\mathrm{Hilb}\mathbf{A}^2 \;\simeq\; \mathrm{Coh}^{tf}_{\mathrm{GL}_1}(\mathbf{P}^2, \mathbf{P}^1).$$

2.12. This construction generalises to arbitrary GL_r, where we can define

$$\mathrm{Hilb}_{\mathrm{GL}_r}\mathbf{A}^2 \;:=\; \mathrm{Coh}^{tf}_{\mathrm{GL}_r}(\mathbf{P}^2, \mathbf{P}^1)$$

as the stack of rank r torsion free sheaves on $\mathbf{P}^2$ along with a trivialisation on a neighbourhood of $\mathbf{P}^1$. Similarly for SL_r if we also ask for a trivialisation of the determinant bundle of $\mathcal{E}^{\vee\vee}$.

Question 2.13. *Can you define a space* $\mathrm{Hilb}_G(S, D)$ *of "G instantons" for any reductive G and algebraic surface with divisor (S, D)?*

The analogue of Groknowski and Nakajima's result is not that $\mathrm{H}^{\mathrm{BM}}(\mathrm{Hilb}_G\mathbf{A}^2)$ is a vertex algebra $\mathcal{W}(\mathbf{A}^2, G)$, but instead is a module for it (and when $G = \mathrm{GL}_1$ this module happens to be the $\mathcal{W}(\mathbf{A}^2, \mathrm{GL}_1)$ itself). It was proven in [MO12, SV12] that

Theorem 2.14. (Maulik-Okounkov, Shiffmann-Vasserot) *Consider the action on* $\mathrm{Hilb}_{\mathrm{GL}_r}\mathbf{A}^2$ *by* $\mathbf{G}_m^2$ *(by the action on* $\mathbf{A}^2$*) and by the maximal torus* $T \subseteq \mathrm{GL}_r$*. Writing* $\mathbf{T} = T \times \mathbf{G}_m^2$*,*

$$\mathrm{H}^{\mathrm{BM}}_{\bullet,\mathbf{T}}(\mathrm{Hilb}_{\mathrm{GL}_r}\mathbf{A}^2) \otimes_{\mathrm{H}^\bullet(\mathrm{B}\mathbf{T})} \mathrm{Frac}\mathrm{H}^\bullet(\mathrm{B}\mathbf{T})$$

is Verma module of weight λ for the affine W algebra $\mathcal{W}^k(\mathfrak{gl}_r, reg)$ *over the ring* $\mathrm{H}^\bullet(\mathrm{B}\mathbf{T})$.[11]

Likewise, [BFN16i] consider more generally when G has ADE type.

[11] See the references for an explicit formula for λ and k.

Question 2.15. *When* $G = \mathrm{SL}_2$*, this gives an action of the Virasoro vertex algebra* $\mathrm{Vir}_c = \mathcal{W}^k(\mathfrak{sl}_2, e)$ *on the rank two space of* S^4*. What is the relation between this and Virasoro contraints?*

Question 2.16. *In the recent paper* [HMMS]*, what should be the algebra of modes of a W algebra* $\mathcal{W}(G, S)$ *was constructed for a large class of* G *and* S*. Can the associated vertex algebra be reconstructed?*

2.17. Swapping G with its Langlands dual gives an isomorphism on affine vertex algebras, due to Feigin and Frenkel:

$$\begin{array}{ccccc} \mathcal{N}=2\ 6d\ \text{SCFT} & & \mathcal{T}[G] & \longleftarrow\!\!-\!-\!-\!-\!-\!\!\longrightarrow & \mathcal{T}[G^L] \\ \Big\downarrow \textstyle\int_{\mathbf{A}^2} & & \Big\downarrow \textstyle\int_{\mathbf{A}^2} & & \Big\downarrow \textstyle\int_{\mathbf{A}^2} \\ \text{Vertex algebras} & & \mathcal{W}^k(\mathfrak{g}, \text{reg}) & \overset{\text{[FF91]}}{=\!=\!=} & \mathcal{W}^{k^L}(\mathfrak{g}^L, \text{reg}) \end{array}$$

For instance, this says that two Virasoros at Langlands dual levels k, k^L have the same central charge, i.e. are isomorphic.

Question 2.18. *What happens to the rest of the AGT picture upon exchanging* G *ang* G^L*?*

2.19. **Ending notes.** A very interesting subject is *quantum AGT*. On a physics level, we begin with a $7d$ theory $\mathcal{T}_q[G]$ and apply the same operations. In particular, this predicts a "q deformed" affine W algebra and AGT correspondence, which has been studied by Frenkel, Reshitikin, Agnagic, Okounkov, and others, but the above story is less developed.

References

[AGT10] Alday, L.F., Gaiotto, D. and Tachikawa, Y., 2010. Liouville correlation functions from four-dimensional gauge theories. Letters in Mathematical Physics, 91(2), pp.167-197.

[BD91] Beilinson, A. and Drinfeld, V., 1991. Quantization of Hitchin's integrable system and Hecke eigensheaves.

[BDG15] Bullimore, M., Dimofte, T. and Gaiotto, D., 2015. The Coulomb Branch of 3d $\mathcal{N} = 4$ Theories. arXiv e-prints. arXiv preprint arXiv:1503.04817.

[BFN16i] Braverman, Alexander, Michael Finkelberg, and Hiraku Nakajima. "Coulomb branches of $3d$ $\mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian (with appendices by Alexander Braverman, Michael Finkelberg, Joel Kamnitzer, Ryosuke Kodera, Hiraku Nakajima, Ben Webster, and Alex Weekes)." arXiv preprint arXiv:1604.03625 (2016).

[BFN16ii] Braverman, A., Finkelberg, M. and Nakajima, H., 2016. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. arXiv preprint arXiv:1601.03586.

[FBZ04] Frenkel, E. and Ben-Zvi, D., 2004. *Vertex algebras and algebraic curves* (No. 88). American Mathematical Soc..

[FF91] Feigin, B. and Frenkel, E., 1991. Duality in W-algebras. International Mathematics Research Notices, 1991(6), pp.75-82.

[Gr95] Grojnowski, I., 1995. Instantons and affine algebras I: the Hilbert scheme and vertex operators. arXiv preprint alg-geom/9506020.

[HMMS] Hausel, T., Mellit, A., Minets, A. and Schiffmann, O., 2022. $P = W$ via H_2. arXiv preprint arXiv:2209.05429.

[MO12] Maulik, D. and Okounkov, A., 2012. Quantum groups and quantum cohomology. arXiv preprint arXiv:1211.1287.

[Na99] Nakajima, H., 1999. Lectures on Hilbert schemes of points on surfaces (No. 18). American Mathematical Soc..

[So20] Soibelman, Y. Cohomological Hall algebras, Nekrasov instantons, and Coulomb branches of $3d$ $\mathcal{N} = 4$ gauge theories. Notes for a talk at Maimi Winter School, 2020.

[SV12] Schiffmann, O. and Vasserot, E., 2013. Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on A 2. Publications mathématiques de l'IHÉS, 118(1), pp.213-342.

[SW96] Seiberg, N. and Witten, E., 1996. Gauge dynamics and compactification to three dimensions. arXiv preprint hep-th/9607163.

[Te21] Teleman, C., 2021. The rôle of Coulomb branches in 2D gauge theory. Journal of the European Mathematical Society, 23(11), pp.3497-3520.

[KW06] Kapustin, A. and Witten, E., 2006. Electric-magnetic duality and the geometric Langlands program. arXiv preprint hep-th/0604151.

[Wi06] Witten, E., 2009. Geometric Langlands from six dimensions. arXiv preprint arXiv:0905.2720.