# DOUBLE POISSON VERTEX ALGEBRAS

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Abstract. In this talk, we introduce the theory of double Poisson algebras, developped by Van den Bergh, and its vertex analogue, developped by de Sole, Kac and Valeri. As an application, we present the induced Poisson (vertex) structure that one can put on representation moduli spaces of a double Poisson (vertex) algebra.

## 1. Introduction

A motivation for associative Poisson algebra is the fact that they give a framework for Hamiltonian equations, which are very important objects coming from mathematical physics. Let $X$ be a manifold and $M := \mathsf{T}^* X$ be its cotangent bundle. It is a symplectic manifold, hence its coordinate ring $\mathscr{O}(M)$ is a Poisson algebra, with Poisson bracket $\{\bullet, \bullet\}$. Fix $h \in \mathscr{O}(M)$ a function, called Hamiltonian function in this context, and denote by $H$ the corresponding Hamiltonian vector field. An integral curve $\gamma : I \to M$ of $H$ is characterized by the following property: for all $f \in \mathscr{O}(M)$ and $t \in I$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma(t)) = \{h, f\} \circ \gamma(t).$$

There are several generalizations for Poisson brackets. If one tries to replace $\mathscr{O}(M)$ by a non-commutative ring $R$, to study noncommutative Hamiltonian ordinary differential equations, one can use the formalism of double Poisson bracket introduced by Van den Bergh in [VdB08]. Double Poisson vertex algebras are introduced in [DSKV15] by de Sole, Kac and Valeri as a formalism for noncommutative partial differential equations. In this talk, we give an introduction to these theories and, as an application, we present the construction of the induced Poisson (vertex) structure on the moduli spaces claffisying the representations of the double Poisson (vertex) algebra.

**Convention.** In this talk, all algebras are assumed to have a unit element.

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Contents



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## 2. Double Poisson associative algebras

Let $R$ be an associative $\mathbf{C}$-algebra. Denote by $\mu : R \otimes_{\mathbf{C}} R \to R$ the multiplication map and $\sigma : R \otimes_{\mathbf{C}} R \to R \otimes_{\mathbf{C}} R$ the permutation map:

$$\sigma(a \otimes b) \coloneqq b \otimes a, \quad a, b \in R.$$

There is a $R$-bimodule structure on $R \otimes_{\mathbf{C}} R$ given by the formula

$$a(b \otimes c)d = ab \otimes cd, \quad a, b, c, d \in R.$$

Denote by $\odot$ the natural product on $R \otimes_{\mathbf{C}} R^{\mathsf{op}}$, defined explicitely as:

$$(a \otimes b) \odot (c \otimes d) \coloneqq ac \otimes db, \quad a, b, c, d \in R.$$

**Definition 2.1** ([VdB08])**.** *A **double Poisson bracket** on $R$ is a linear map*

$$\{\{\bullet, \bullet\}\} : R \otimes_{\mathbf{C}} R \longrightarrow R \otimes_{\mathbf{C}} R$$

*such that the following axioms hold. Set $a, b, c \in R$.*

(1) *Skew-symmetry:* $\{\{a, b\}\} = -\sigma(\{\{b, a\}\})$.
(2) *Left Leibniz rule:* $\{\{a, bc\}\} = \{\{a, b\}\}\, c + b\, \{\{a, c\}\}$.
(3) *Jacobi identity:*

$$\{\{a, \{\{b, c\}\}\}\}_{\mathsf{L}} - \{\{b, \{\{a, c\}\}\}\}_{\mathsf{R}} = \{\{\{\{a, b\}\}, c\}\}_{\mathsf{L}},$$

*whith the following notations for elements in $R \otimes_{\mathbf{C}} R \otimes_{\mathbf{C}} R$:*

$$\{\{a, b \otimes c\}\}_{\mathsf{L}} \coloneqq \{\{a, b\}\} \otimes c,$$
$$\{\{b, a \otimes c\}\}_{\mathsf{R}} \coloneqq a \otimes \{\{b, c\}\},$$
$$\{\{a \otimes b, c\}\}_{\mathsf{L}} \coloneqq \sum_{t \in T} u_t \otimes b \otimes v_t \quad \text{where} \quad \{\{a, c\}\} = \sum_{t \in T} u_t \otimes v_t.$$

*Equipped with $\{\{\bullet, \bullet\}\}$, the algebra $R$ is called a **double Poisson algebra**.*

*Example* 2.2. Set $R \coloneqq \mathbf{C}\langle x_1, \ldots, x_n\rangle$, the algebra of noncommutative polynomials. We define the noncommutative partial derivative $\frac{\partial}{\partial x_j} : R \to R$ by the formula:

$$\frac{\partial}{\partial x_j}(x_{i_1} \cdots x_{i_k}) = \sum_{\ell=1}^{k} \delta_{i_\ell = j} x_{i_1} \cdots x_{i_{\ell-1}} \otimes x_{i_{\ell+1}} \cdots x_{i_k}.$$

Poisson double brackets on $R$ are all of the form

$$\{\{f, g\}\} = \sum_{1 \leqslant i,j \leqslant n} \frac{\partial g}{\partial x_j} \odot \{\{x_i, x_j\}\} \odot \sigma\left(\frac{\partial f}{\partial x_i}\right)$$

where the quantities $\{\{x_i, x_j\}\} \in R \otimes R$, $1 \leqslant i, j \leqslant n$, verify the skew-symmetry axiom and the Jacobi identity.

*Example* 2.3. Let $R \coloneqq \mathbf{C}\langle x\rangle$ be the free algebra spanned by one element. Up to automorphism, there are two nontrivial possibilities for a double Poisson bracket on this algebra:

$$\{\{x, x\}\} = x \otimes 1 - 1 \otimes x \quad \text{or} \quad \{\{x, x\}\} = x^2 \otimes x - x \otimes x^2.$$

Denote by $\overline{R}$ the quotient $R/[R, R]$ and call trace the quotient map $\mathrm{tr} : R \to \overline{R}$. For $a, b \in R$, set:

$$\begin{aligned} \{a, b\} &\coloneqq \mu(\{\{a, b\}\}), \\ \{\mathrm{tr}(a), b\} &\coloneqq \{a, b\} = \mu(\{\{a, b\}\}), \\ \{\mathrm{tr}(a), \mathrm{tr}(b)\} &\coloneqq \mathrm{tr}(\{a, b\}) = \mathrm{tr}(\mu(\{\{a, b\}\})). \end{aligned} \tag{2.1}$$

**Lemma 2.4** ([VdB08])**.** *These formulae induce well-defined maps*

$$\{\bullet, \bullet\} : \overline{R} \otimes_{\mathbf{C}} R \to R \quad \text{and} \quad \{\bullet, \bullet\} : \overline{R} \otimes_{\mathbf{C}} \overline{R} \to \overline{R}.$$

*Proof.* Check the formulae:

$$\{[a,b],c\} = 0 \quad \text{and} \quad \{a,[b,c]\} = [\{a,b\},c] + [b,\{a,c\}].$$

The lemma follows. □

**Proposition 2.5** ([VdB08])**.** *The pair $\left(\overline{R}, \{\bullet,\bullet\}\right)$ is a Lie algebra which acts on $R$ by derivation according to the formula (2.1).*

*Remark* 2.6. A noncommutative hamiltonian ordinary differential equation is an equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \{\mathrm{tr}(h), x\}$$

where $h, x(t) \in R$.

From now, assume that $R$ is finitely generated as a $\mathbf{C}$-algebra, with generators $x_1, \dots, x_n$. Fix $d \geqslant 1$ an integer. Denote by $Y_d$ the subscheme of $\mathrm{Mat}_d(\mathbf{C})^{\times n}$ which encodes $\mathrm{Hom}_{\mathsf{Alg}}(R, \mathrm{Mat}_d(\mathbf{C}))$ and denote by $R_d$ its coordinate ring.

There is a natural map

$$R \longrightarrow \mathrm{Mat}_d(R_d), \quad a \longmapsto (a_{i,j})_{1 \leqslant i,j \leqslant d}$$

where for all $\xi \in Y_d$, the formula $a_{i,j}(\xi) \coloneqq \xi_{i,j}(a)$ defines an element of $R_d$. This map induces another one:

$$\phi_d : \overline{R} \longrightarrow R_d, \quad \mathrm{tr}(x) \longmapsto \sum_{i=1}^{d} x_{i,i}.$$

*Remark* 2.7. There is an action of $\mathrm{GL}_d(\mathbf{C})$ defined on $R_d$ by:

$$g \cdot a_{i,j} \coloneqq \sum_{h,k=1}^{d} [g^{-1}]_{i,h} a_{h,k} g_{k,j}, \quad a \in R, \quad g \in \mathrm{GL}_d(\mathbf{C}), \quad 1 \leqslant i,j \leqslant d,$$

it is in fact induced by the natural action of $\mathrm{GL}_d(\mathbf{C})$ on $Y_d$. The image of $\overline{R}$ by $\phi_d$ lies into the subalgebra of invariants, ${R_d}^{\mathrm{GL}_d(\mathbf{C})}$.

**Proposition 2.8** ([VdB08])**.** *The algebra $R_d = \mathscr{O}(\mathrm{Hom}_{\mathsf{Alg}}(R, \mathrm{Mat}_d(\mathbf{C})))$ can be equipped with a Poisson bracket defined by the formula*

$$\{a_{i,j}, b_{h,k}\} \coloneqq \sum_{t \in T} [c_t]_{h,j} [d_t]_{i,k} \quad \textit{where} \quad \{\{a,b\}\} = \sum_{t \in T} c_t \otimes d_t,$$

*for $a, b \in R$. Then, $\phi_d : \overline{R} \to R_d$ is a Lie algebra morphism.*

*Example* 2.9. Let $R \coloneqq \mathbf{C}\langle x \rangle$. One has $Y_d = \mathrm{Mat}_d(\mathbf{C})$ and

$$R_d = \mathscr{O}(\mathrm{Mat}_d(\mathbf{C})) = \mathbf{C}[u_{i,j} \mid 1 \leqslant i,j \leqslant d].$$

If $R$ is equipped with the Poisson bracket given by

$$\{\{x,x\}\} = x \otimes 1 - 1 \otimes x,$$

then $R_d$ is equipped with its usual Poisson structure:

$$\{u_{i,j}, u_{h,k}\} = \delta_{i=k} u_{h,j} - \delta_{h=j} u_{i,k}.$$

This construction can be adapted to the path algebra $\mathbf{C}\langle Q \rangle$ of a quiver $Q \coloneqq (I, A, s, e)$. Denote by $1_i$, $i \in I$, the orthogonal idempotent elements corresponding to the empty path at the vertex $i$. The unit of the algebra is $1 = \sum_{i\in I} 1_i$. If $\mathbf{d} \coloneqq (d_i)_{i\in I}$ is a dimension vector, define

$$\begin{aligned} Y_{\mathbf{d}} &\coloneqq \operatorname{Hom}_{(\bigoplus_{i\in I}\mathbf{C}1_i)\text{-}\mathsf{Alg}}\left(\mathbf{C}\langle Q\rangle, \operatorname{End}_{\mathbf{C}}\left(\bigoplus_{i\in I}\mathbf{C}^{d_i}\right)\right) \\ &\cong \bigoplus_{a\in A}\operatorname{Mat}_{d_{s(a)},d_{e(a)}}(\mathbf{C}) \end{aligned}$$

the moduli space of representations of $Q$ of dimension vector $\mathbf{d}$.

Take $\overline{Q}$ the double quiver associated to $Q$ and denote by $\overline{Y_{\mathbf{d}}}$ the corresponding moduli space. To all edges $a \in A$ one adds an edge in the opposite direction, denoted by $a^*$. One can define a double Poisson bracket on the generators of $\mathbf{C}\left\langle\overline{Q}\right\rangle$ by:

$$\{\{a, a^*\}\} \coloneqq 1_{s(a)} \otimes 1_{e(a)},$$

with all the other double brackets assumed to be 0.

*Remark* 2.10. One can identify $\overline{Y_{\mathbf{d}}}$ with the cotangent bundle $\mathsf{T}^*Y_{\mathbf{d}}$. Then, the Poisson structure induced by this double Poisson bracket corresponds exactly to the one induced by the symplectic structure.

## 3. Double Poisson vertex algebra

Let $V$ be an associative $\mathbf{C}$-algebra equipped with a differential $\partial : V \to V$. Denote by $V[\lambda]$ the formal polynomials with variable $\lambda$ and coefficients in $V$. The variable $\lambda$ is written on the left: the formula $\sum_{k\geqslant 0}\lambda^k a_k$ denotes a generic element in $V[\lambda]$. Hence, if $\lambda$ is substituted by some endomorphism of $V$, one gets a well-defined element in $V$.

The multiplication map $\mu$ and the permutation map $\sigma$ naturally extend to formal polynomials, as $\odot$ and the bimodule structure do.

**Definition 3.1** ([DSKV15])**.** *A* ***double Poisson vertex bracket*** *on $V$ is a linear map*

$$\begin{aligned} \{\{\bullet_\lambda\bullet\}\} : V\otimes_{\mathbf{C}} V &\longrightarrow V\otimes_{\mathbf{C}} V[\lambda] \\ a\otimes b &\longmapsto \sum_{k\geqslant 0}\lambda^k\left\{\left\{a_{(k)}b\right\}\right\} \end{aligned}$$

*such that the following axioms hold. Set $a, b, c \in V$.*

1. *Skew-symmetry:* $\{\{a_\lambda b\}\} = -\sigma(\{\{b_{-\lambda-\partial}a\}\})$.
2. *Left sesquilinearity:* $\{\{\partial a_\lambda b\}\} = -\lambda\{\{a_\lambda b\}\}$.
3. *Left Leibniz rule:* $\{\{a_\lambda bc\}\} = \{\{a_\lambda b\}\}c + b\{\{a_\lambda c\}\}$.
4. *Jacobi identity:*

$$\{\{a_\lambda\{\{b_\mu c\}\}\}\}_{\mathsf{L}} - \{\{b_\mu\{\{a_\lambda c\}\}\}\}_{\mathsf{R}} = \{\{\{\{a_\lambda b\}\}_{\lambda+\mu}c\}\}_{\mathsf{L}}.$$

*Equipped with $\{\{\bullet,\bullet\}\}$, the differential algebra $V$ is called a* ***double Poisson vertex algebra****.*

*Example* 3.2. Set $V \coloneqq \mathbf{C}\left\langle x_i^{(p)} \mid 1\leqslant i\leqslant n,\ p\in\mathbf{Z}_{\geqslant 0}\right\rangle$, the algebra of noncommutative differential polynomials, which is equipped with the differential $\partial$ defined by the relations:

$$\partial x_i^{(k)} \coloneqq x_i^{(k)}, \quad 1\leqslant i\leqslant n, \quad k\in\mathbf{Z}_{\geqslant 0}.$$

Define the noncommutative partial derivative $\frac{\partial}{\partial x_j^{(q)}} : V \to V$ by the formula:

$$\frac{\partial}{\partial x_j}(x_{i_1}^{(p_1)}\cdots x_{i_k}^{(p_k)}) = \sum_{\ell=1}^{k}\delta_{i_\ell=j}\delta_{p_\ell=q}x_{i_1}^{(p_1)}\cdots x_{i_{\ell-1}}^{(p_{\ell-1})}\otimes x_{i_{\ell+1}}^{(p_{\ell+1})}\cdots x_{i_k}^{(p_k)}.$$

Poisson double vertex brackets on $V$ are all of the form

$$\{\{f_\lambda g\}\} = \sum_{\substack{1\leqslant i,j\leqslant n\\ p,q\in\mathbf{Z}}} \frac{\partial g}{\partial x_j^{(q)}} \odot (\lambda+\partial)^q \left\{\left\{x_i^{(0)}{}_{(\lambda+\partial)} x_j^{(0)}\right\}\right\}_\to (-\lambda-\partial)^p \odot \sigma\left(\frac{\partial f}{\partial x_i^{(p)}}\right)$$

where the quantities $\left\{\left\{x_i^{(0)}{}_\lambda x_j^{(0)}\right\}\right\}$, $1\leqslant i,j\leqslant n$, verify the skew-symmetry axiom and the Jacobi identity. The notation $\left\{\left\{a_{(\lambda+\partial)}b\right\}\right\}_\to c$ means that $\partial$ has to be apllied to the right:

$$\left\{\left\{a_{(\lambda+\partial)}b\right\}\right\}_\to c \coloneqq \sum_{k\geqslant 0}\sum_{i=0}^{k}\binom{k}{i}\lambda^i \left\{\left\{a_{(k)}b\right\}\right\}\partial^{k-i}c.$$

Denote by $\overline{V}$ the quotient $V/[V,V]$ and call trace the quotient map $\operatorname{tr}: V\to\overline{V}$. Denote by $\widetilde{V}$ the quotient $V/([V,V]+\partial V)$ and call residue the quotient map $\int: V\to\widetilde{V}$. For $a,b\in V$, set:

$$\begin{aligned}\{a_\lambda b\} &\coloneqq \mu(\{\{a_\lambda b\}\}),\\ \{a,b\} &\coloneqq \{a_\lambda b\}\,|_{\lambda=0},\\ \left\{\textstyle\int a, b\right\} &\coloneqq \{a,b\},\\ \left\{\textstyle\int a, \int b\right\} &\coloneqq \textstyle\int\{a,b\}.\end{aligned}$$

**Lemma 3.3** ([DSKV15])**.** *These formulae induce well-defined map*

$$\{\bullet,\bullet\}:\widetilde{V}\otimes_{\mathbf{C}} V\to V \quad \textit{and} \quad \{\bullet,\bullet\}:\widetilde{V}\otimes_{\mathbf{C}}\widetilde{V}\to\widetilde{V}.$$

**Proposition 3.4** ([DSKV15])**.** *The pair* $\left(\widetilde{V},\{\bullet,\bullet\}\right)$ *is a Lie algebra which acts on* $V$ *by derivation and whose action commutes with* $\partial$.

*Remark* 3.5. There is a structure of conformal Lie algebra on $\overline{V}$ equipped with the induced map $\partial$, that is to say there is an induced brackt $\{\bullet_\lambda\bullet\}$ which is skew-symmetric, left sesquilinear and verifies the Jacobi identity. This conformal Lie algebra acts on $R$ by conformal derivations. Quotienting by $\partial\overline{V}$, one gets the action of the Lie algebra $\widetilde{V}$ on $V$.

Let us discuss a vertex analogue to the construction of the induced Poisson structure on the variety of representations. Denote by $V_d$ the commutative differential algebra spanned by the formal elements

$$x_{i,j} \quad \text{for} \quad x\in V, \quad 1\leqslant i,j\leqslant d,$$

with the relations

$$(x+y)_{i,j}=x_{i,j}+y_{i,j}, \quad (zx)_{i,j}=zx_{i,j}, \quad (xy)_{i,j}=\sum_{k=1}^{d}x_{i,k}y_{k,j}, \quad (\partial x)_{i,j}=\partial x_{i,j},$$

for $x,y\in V$, $z\in\mathbf{C}$ and $1\leqslant i,j\leqslant d$.

**Proposition 3.6** ([DSKV15])**.** *The differential algebra* $V_d$ *can be equipped with a Poisson vertex bracket defined by the formula*

$$\{a_{i,j}{}_\lambda b_{h,k}\}\coloneqq\sum_{k\geqslant 0}\lambda^k\sum_{t\in T_k}[c_{k,t}]_{h,j}[d_{k,t}]_{i,k} \quad \textit{where} \quad \left\{\left\{a_{(k)}b\right\}\right\}=\sum_{t\in T_k}c_{k,t}\otimes d_{k,t},$$

*for* $a,b\in V$.

*Example* 3.7. Let $V\coloneqq\mathbf{C}\langle x^{(p)}\mid p\in\mathbf{Z}_{\geqslant 0}\rangle$. One has

$$R_d=\mathbf{C}\left[u_{i,j}^{(p)}\mid 1\leqslant i,j\leqslant d,\ p\in\mathbf{Z}_{\geqslant 0}\right];$$

as a differential algebra, it coincides with $V^c(\mathrm{Mat}_d(\mathbf{C}))$, the universal Poisson vertex algebra at level $c$ associated to the Lie algebra $\mathrm{Mat}_d(\mathbf{C})$ and the trace bilinear form. If $V$ is equipped with the double Poisson vertex bracket given by

$$\{\{x_\lambda x\}\} = x \otimes 1 - x \otimes 1 + \lambda c(1 \otimes 1),$$

where $c \in \mathbf{C}$ is a fixed constant called level, then $V_d$ is equipped with its usual Poisson vertex bracket:

$$\{u_{i,j\,\lambda} u_{h,k}\} = \delta_{i=k} u_{h,j} - \delta_{h=j} u_{i,k} + \lambda c \delta_{i=k} \delta_{j=h}.$$

## References

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