# Arc spaces and Poisson vertex algebras

Workshop on Hall algebras and vertex algebras in enumerative geometry

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## Introduction

In this talk, I will firstly introduce the notion of **Arc spaces** in geometry. These spaces are interesting because their function ring are commutative vertex algebras. Also, this is a way to relate geometry to the theory of vertex alebras, by studying the geometry of a scheme, we can deduce properties for the geometry of arc spaces and then deduce properties for its vertex algebra.

I will secondly introduce **Poisson vertex algebra**, that is a new structure for commutative vertex algebras. There is essentially two examples of Poisson vertex alegras arising in nature, the first one is for the ring of functions of the arc space of a Poisson scheme, these are endowed with a Poisson vertex algebra structure. The second one comes from any vertex algebras. All of them have a canonical filtration called the **Li filtration**. Its associated graded algebra is a Poisson vertex algebra. We can then use these examples to define the singular support of a vertex algebra and prove some new properties on the algebra.

The reference is [AM21], chapters 1 and 4.

We work over $\mathbf{C}$.

## Arc spaces

**Definition.** *Let $A$ be an algebra, $X := \operatorname{Spec}(A)$ the affine scheme. The ring $\mathcal{J}_\infty A$ of functions of the* ***arc space*** *of $X$ is the free differential algebra spanned by $A$. We set $\mathcal{J}_\infty X := \operatorname{Spec}(\mathcal{J}_\infty A)$, it's a space over $X$. Recall that a* ***differential algebra*** *is a commutative algebra $A$ with a derivation $\partial : A \to A$, that is, a linear map satisfying $\partial ab = a\partial b + b\partial a$.*

The differential algebras are actually the same as commutative vertex algebras (cf. the talk from Robert), thus, the space of functions of arc spaces are commutative vertex algebras.

The globalisation of arc spaces for arbitrary schemes induces a sheaf of commutative vertex algebras.

On the algebra level, we have a canonical filtration $A \subset \mathcal{J}_1 A \subset \cdots \subset \mathcal{J}_\infty A$ where $\mathcal{J}_n A$ is the subalgebra of $\mathcal{J}_\infty A$ spanned by the $\partial^i a$ for $a \in A$ and $0 \leq i \leq n$. $\mathcal{J}_n X := \operatorname{Spec}(\mathcal{J}_n A)$ is the $n$-th order **jet space** of $X$, it comes with truncation functions : $\pi_{n,m} : \mathcal{J}_n X \to \mathcal{J}_m X$.

Of course, the jet spaces constructions are functorial, if $f : X \to Y$ is a morphism of schemes, it induces a morphism $\mathcal{J}_n f : \mathcal{J}_n X \to \mathcal{J}_n Y$ on jet spaces.

We define the infinitesimal disks $\mathbb{D}_n := \operatorname{Spec}(\mathbf{C}[z]/z^n)$ and $\mathbb{D}_\infty := \operatorname{Spec}(\mathbf{C}[[z]])$. They are pointed by the point $0 : \operatorname{Spec}(\mathbf{C}) \to \mathbb{D}_n$. We have these universal properties naturally in any schemes $X$ of finite type and arbitrary $Z$ :

$$\operatorname{Hom}_{Sch}(Z, \mathcal{J}_n X) \simeq \operatorname{Hom}_{Sch}(Z \times \mathbb{D}_n, X),$$

$$\operatorname{Hom}_{Sch}(Z, \mathcal{J}_\infty X) \simeq \operatorname{Hom}_{fSch}(Z \hat{\times} \mathbb{D}_\infty, X),$$

where the completion is taken across $Z \times \{0\}$.

Because of the universal property we have $\mathcal{J}_\infty(X \times X) \simeq \mathcal{J}_\infty X \times \mathcal{J}_\infty X$ and $\mathcal{J}_\infty(A \otimes A) \simeq \mathcal{J}_\infty A \otimes \mathcal{J}_\infty A$.

## Loop space

The arc space of a scheme $X$ classifies the $\mathbb{D}_\infty$-points of $X$. The **loop space** of $X$ will be a space that classifies intuitively $S^1 = (\mathbb{D}_\infty \setminus \{0\})$-points of $X$, it actually clasifies $\mathbb{C}((z))$-points.

**Definition.** *For $X$ an affine scheme of finite type, there exists an ind-scheme $\mathscr{L}X$ that classifies $\mathbb{C}((z))$-points of $X$ :*

$$\mathrm{Hom}_{Sch}(\mathrm{Spec}(\mathbb{C}((z))), X) \simeq \mathrm{Hom}_{indSch}(\mathrm{Spec}(\mathbb{C}), \mathscr{L}X).$$

*We can embed the arc space in the loop space : $\mathscr{J}_\infty X \to \mathscr{L}X$.*

What is interesting about loop space is this statement : if $X$ is an affine scheme, then the category of vertex $\mathscr{O}(\mathscr{J}_\infty X)$-modules is equivalent to the category of smooth $\mathscr{O}(\mathscr{L}X)$-modules. (The ring $\mathscr{O}(\mathscr{L}X)$ is a topological ring because $\mathscr{L}X$ is an ind-scheme).

## Poisson vertex algebra

Recall that a **vertex algebra** is the data of a vector space $V$ with a representation of itself in its fields $Y : V \to \mathscr{F}ields(V)$, a vacuum vector $|0\rangle \in V$ and a translation operator $T : V \to V$ satisfying some axioms.
When it is commutative, $V$ is a differential algebra, namely the structure is given by $ab := a_{(-1)}b$ with unit $|0\rangle$ and $\partial a := Ta$ and we have $a_{(n)}b = 0$ for $n \geq 0$.

**Definition.** *A **Poisson vertex algebra** $V$ is a commutative vertex algebra together with a $\lambda$-bracket :*

$$\{\bullet_\lambda \bullet\} : V \otimes V \to V[\lambda]$$

*which satisfies :*

- *(**Sesquilinearity**) : $\{(\partial a)_\lambda b\} = (\lambda + \partial)\{a_\lambda b\}$,*
- *(**Skewsymmetry**) : $\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\}$,*
- *(**Jacobi identity**) : $\left\{a_\lambda \left\{b_\mu c\right\}\right\} - \left\{b_\mu \{a_\lambda c\}\right\} = \left\{\{a_\lambda b\}_\mu c\right\}$,*
- *((left) **Leibniz rule**) : $\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\}$.*

*We choose to write $\{a_\lambda b\} = \sum_{n\geq 0} a_{(n)} b \frac{\lambda^n}{n!}$ which is consistent with the old notation because $a_{(n)}b = 0$ for $n \geq 0$ in the old notation as $V$ is commutative.*

### Poisson vertex algebra structure on the arc space

If $X = \mathrm{Spec}(A)$ is an affine Poisson scheme, that means that $A$ is given with a Poisson bracket $\{\bullet, \bullet\} : A \otimes A \to A$ satisfying anticommutativity, the Jacobi identity and the Leibniz rule.
The Poisson bracket $A \otimes A \to A[\lambda]$ (that sends $a, b$ to $\{a, b\}\lambda^0$) induces a bracket $\mathscr{J}_\infty A \otimes \mathscr{J}_\infty A \to \mathscr{J}_\infty A[\lambda]$ which is a Poisson vertex algebra bracket. Thus, $\mathscr{J}_\infty A$ is a Poisson vertex algebra.

### Associated Poisson algebra and associated variety of a vertex algebra

Let $V$ be a vertex algebra (non necessarly commutative), a set $\{a^i\}_{i\in I} \subset V$ of vectors of $V$ is called a set of **strong generators** if it generates $V$ in the sense that each $v \in V$ is of the form $a^{i_1}_{(-n_1-1)} \cdots a^{i_k}_{(-n_k-1)} |0\rangle$ for $i_1, \dots, i_k \in I$ and $n_1, \dots, n_k \geq 0$. The set $V$ is a set of strong generators.
Given such a set, we can filtrate $V$ by setting

$$F^p V := \mathrm{Vect}\{a^{i_1}_{(-n_1-1)} \cdots a^{i_k}_{(-n_k-1)} |0\rangle \mid n_1 + \cdots + n_k \geq p\}.$$

We have $V = F^0 V \supset F^1 V \supset \dots$. This filtration doesn't depend of the choice of the set of strong generators and is called the **Li filtration**.

Because we have a filtration, we can take its associated graded algebra :

$$\operatorname{gr}^F V := \bigoplus_{p\geq 0} F^p V / F^{p+1} V.$$

This vector space is actually endowed with a Poisson vertex algebra structure given by $ab := a_{(-1)}b$, $\partial a := Ta$ and $a_{(n)}b$ (new notation) $:= a_{(n)}b$ (old notation), for $n \geq 0$.
Denote by $R_V$ the first graduation $V/F^1V$. Then $R_V$ is a Poisson algebra by $\{a, b\} := a_{(0)}b$. $R_V$ is called the **Zhu $C_2$-algebra** and spans $\operatorname{gr}^F V$ as a differential algebra. We can finally define the **associated scheme** and the **associated variety** of $V$ :

$$\tilde{X}_V := \operatorname{Spec}(R_V) \text{ and } X_V := \left(\tilde{X}_V\right)_{\text{red}}.$$

Because $R_V$ generates $\operatorname{gr}^F V$ as a differential algebra, we have a surjective map of differential algebras :

$$\mathscr{J}_\infty \tilde{X}_V \to \operatorname{gr}^F V.$$

Actually, it is a Poisson vertex algebra morphism. We define the **singular support** of $V$ :

$$\operatorname{SS}(V) := \operatorname{Spec}\left(\operatorname{gr}^F V\right) \subset \mathscr{J}_\infty \tilde{X}_V.$$

We can then study the singular support of $V$ to deduce properties on $V$, for instance, $\operatorname{SS}(V) = \mathscr{J}_\infty \tilde{X}_V$ and $R_V$ is a polynomial ring iff $V$ satisfies the Poincaré-Birkhoff-Witt theorem, that is, there exists a set of strong generators $\{a_i\}_{i\in I}$, $I$ ordered, such that $a^{i_1}_{(-n_1-1)} \cdots a^{i_k}_{(-n_k-1)} |0\rangle$ for $i_1 \leq i_2 \leq \cdots \leq i_k$ and $n_l \leq n_{l+1}$ if $i_l = i_{l+1}$ for a basis of $V$.

## The lisse condition

Assume in the sequel that $V$ is strongly generated by a finite set of elements, that means that $X_V$ is of finite type.

**Definition.** *A vertex algebra $V$ is called **lisse** (french for "smooth") if* $\dim X_V = 0$*, that is, if $R_V$ is finite dimensional.*

We can relate this condition to the singular support, $\dim \operatorname{SS}(V) = 0$ iff $V$ is lisse.
The lisse condition implies for instance that if $V$ is generated by the $\{a_i\}_{i\in I}$, then the $\partial^n a_i \in \mathscr{O}(\mathscr{L}\tilde{X}_V)$ are nilpotent.
Another implication is if $V$ is conformal and conical, then, if $V$ is lisse, every $\mathbf{Z}$-graded simple vertex $V$-module is $\mathbf{Z}_{>0}$-graded.
A generalization of the lisse condition is the quasi-lisse condition. We say that $V$ is quasi-lisse if $\tilde{X}_V$ have finitely many symplectic leaves.

# References

[AM21] Tomoyuki Arakawa and Anne Moreau. Arc spaces and vertex algebras, 2021.