QUANTIZATION OF VERTEX ALGEBRAS

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We follow [EK98] in order to introduce quantum vertex operator algebras. We will start by recalling vertex operator algebras, then define braided vertex operator algebras as a first attempt at naive quantization and then add an additional axiom in order to guarantee "associativity". We then present basic properties of these quantum vertex algebras and run through a few examples.

Notation: Throughout $\Bbbk$ will denote a field of characteristic zero. If V is a topologically free $\Bbbk[[h]]$ module $V((z))$ denotes the power series $\sum v_n z^n$ with $v_n \to 0$ as $n \to -\infty$.

1. Setup

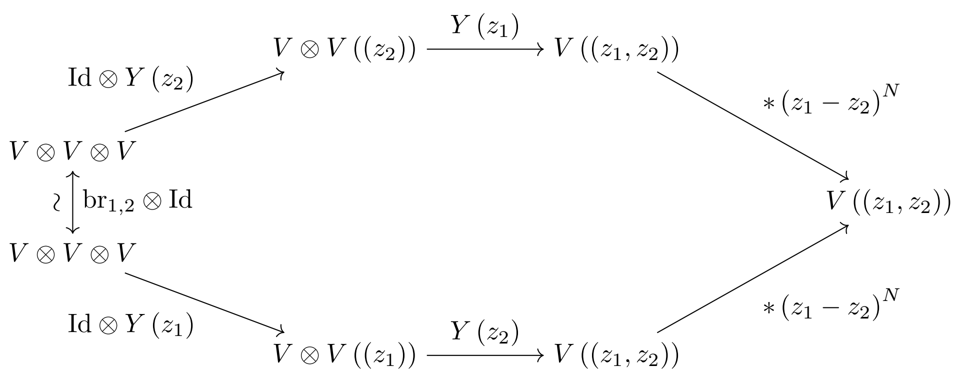
We want to quantize the following:

Definition 1.1 (Vertex operator algebra)**.** A vertex operator algebra consists of

- $V \in \mathrm{Mod}(\Bbbk)$
- a linear map

$$Y : V \otimes V \to V((z))$$
$$v \otimes w \mapsto Y(z)(v \otimes w)$$

- a linear operator $T : V \to V$ (shift/Sugawara operator)
- a vector $\Omega \in V$ (the vacuum vector) s.t.:
  (1) locality: $\forall v, w \in V \exists N \geq 0$ such that the following diagram commutes



  (2) $T\Omega = 0$ and $\frac{d}{dz} Y(z) = TY(z) - Y(z)(\mathrm{Id} \otimes T)$.
  (3) $Y(z)(\Omega \otimes v) = v$ and $Y(z)(v \otimes \Omega)$ is regular at 0 and $Y(0)(v \otimes \Omega) = v$

Remark 1.2. Observe the following:

- Y and Ω fix T.
- Sometimes one denotes $Y(z)(v \otimes \_) \in \mathrm{End}(V)((z))$ by $Y(v, z)$.

In order to understand the settings and examples we also recall the following notion, as we will also quantize these settings.

Definition 1.3. A derivation of a vertex operator algebra is a linear map $X : V \to V$ such that $XY(z) = Y(z)(X \otimes \mathrm{Id} + \mathrm{Id} \otimes X)$

Remark 1.4. For a derivation of a vertex operator algebra we have $[T, X] = 0$ and $X\Omega = 0$.

2. Braided vertex operator algebras

As with monoidal structures, the first step towards quantization is to consider braided vertex operator algebras, these consist of a very similar structure. To do this we attempt to generalize the locality to $\mathbb{k}[[q]]$-modules such that after reduction to degree zero we get usual locality. However, we will encounter certain deficiencies of braided vertex operator algebras that we will fix by enforcing associativity later.

Definition 2.1. A braided vertex operator algebra over $\mathbb{k}[[q]]$ consists of:

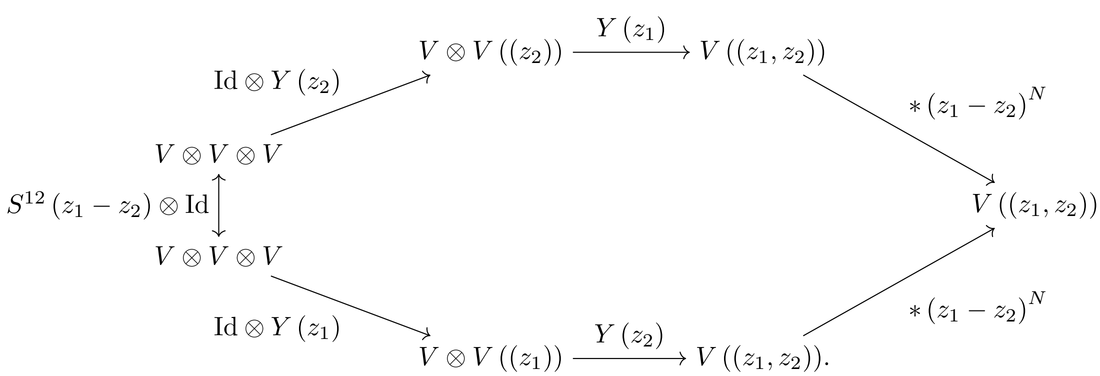
- a topologically free $\mathbb{k}[[q]]$-module V.
- a linear map

$$\begin{aligned} Y : V \otimes V &\to V((z)) \\ v \otimes w &\mapsto Y(z)(v \otimes w). \end{aligned}$$

- $T : V \to V$ (the Sugawara operator)
- $\omega \in V$ (the vacuum vector)
- a linear map $S : V \otimes V \to V \otimes V \otimes \mathbb{k}((z))$ s.t.
  - $S = 1 + O(q)$
  - $[T \otimes \mathrm{Id}, S(z)] = -\frac{dS}{dz}$
  - $S^{12}(z) S^{13}(z+w) S^{23}(w) = S^{23}(w) S^{13}(z+w) S^{12}(z)$ (quantum Yang-Baxter equation).
  - $S^{21}(z) = \left(S^{12}\right)^{-1}(-z)$ (unitarity).

Such that the following holds:

(1) S-locality: $\forall v, w \in V$, $\forall M \in \mathbb{N}$ $\exists N \geq 0$ such that the following diagram commutes:



(2) $T\Omega = 0$ and $\frac{d}{dz} Y(z) = TY(z) - Y(z)(\mathrm{Id} \otimes T)$.
(3) $Y(z)(\Omega \otimes v) = v$ and $Y(z)(v \otimes \Omega)$ is regular at 0 and $Y(0)(v \otimes \Omega) = v$.

Where we take all tensor products in the q-adically complete sense.

In contrast to ordinary vertex operator algebras braided vertex operator algebras are not in general associative. However, the suffice the following quasi-associativity.

Proposition 2.2. *The map Y satisfies quasi-associativity:*

$$Y(z)(\mathrm{Id} \otimes Y(w)) S^{23}(w) S^{13}(z) = Y(w) S(w) (Y(z-w) \otimes \mathrm{Id}).$$

Definition 2.3. A linear map $X : V \to V$ is a derivation of a braided vertex operating algebra V if $XY(z) = Y(z)(X \otimes \mathrm{Id} + \mathrm{Id} \otimes X)$ and $[X \otimes \mathrm{Id} + \mathrm{Id} \otimes X, S(z)] = 0$.

Remark 2.4. As in the classical case one gets that for a derivation X of V we have $[T, X] = 0$, $X\Omega = 0$ and that T is a derivation.

3. Quantum vertex operator algebras

In order to fix the deficiency of a braided vertex operator algebras regarding associativity we define quantum vertex operator algebras to "suffice associativity". Then we will verify a few properties of these quantum vertex operator algebras in order to see that we can indeed work with them in a similar way to ordinary vertex operators algebras and that the imposed associativity, locality and untarity are trivial for non-degenerate quantum vertex operator algebras.

Definition 3.1. A braided vertex operator algebra is a quantum vertex operator algebra if the following hexagon relation is satisfied:

$$S(w)(\mathrm{Id}\otimes Y(w)) = (Y(u)\otimes \mathrm{Id})\, S^{23}(w)\, S^{13}(u+w)\,. \tag{1}$$

Proposition 3.2. *In a quantum vertex operator algebra we have the following associativity:*

$$Y(z)(\mathrm{Id}\otimes Y(w)) = Y(w)(Y(z-w)\otimes \mathrm{Id})\,. \tag{2}$$

Corollary 3.3. *For any v, w in a quantum vertex operator algebra V and any integer n there exists a unique $y_n \in V$ such that*

$$\mathrm{Res}_{z=w}(z-w)^n\, Y(z)(\mathrm{Id}\otimes Y(w))(v\otimes w\otimes u) = Y(w)(y_n\otimes u)\,.$$

The above Corollary 3.3 implies the existence of the operator product expansion:

$$Y(v,z)\, Y(u,w) = \sum_n (z-w)^{-n-1}\, Y(y_n, w)\,.$$

3.1. Quasiclassical vertex operator algebras.

We will now consider the quasiclassical limits of quantum vertex operator algebras when $q\to 0$ in order to see which vertex operator algebras have a chance to be quantized and which do not. It turns out that one crucial property of classical limits of quantum vertex operator algebras is a shadow of the braiding. Vertex operator algebras which admit these shadows are called a quasiclassical vertex operator algebra.

Definition 3.4. Let V be a vertex operator algebra. A classical r-matrix on V is a linear map $s(z): V\otimes V \to V\otimes V\otimes \Bbbk((z))$ such that the following hold:

(1) The classical Yang-Baxter equation with spectral parameter:
$$\left[s^{12}(z_1-z_2), s^{13}(z_1-z_3)\right] + \left[s^{12}\left(z_1-z_2, s^{23}(z_2-z_3)\right)\right] + \left[s^{13}(z_1-z_3), s^{23}(z_2-z_3)\right] = 0.$$
(2) The unitarity condition:
$$s^{21}(-z) = -z(z)\,.$$
(3) The shift condition:
$$[T\otimes \mathrm{Id}, s] = -\frac{ds}{dz}.$$
(4) The hexagon identity:
$$s(w)(Y(u)\otimes \mathrm{Id}) = (Y(u)\otimes \mathrm{Id})\left(s^{23}(u) + s^{13}(u+w)\right).$$

A vertex operator algebra equipped with a classical r-matrix is called a quasiclassical vertex operator algebra.

As mentioned above we have for a V a quantum vertex operator algebra a quasiclassical structure on $V^0 = V/qV$ by considering $S(z) = \mathrm{Id} + qs(z) + O\left(q^2\right)$ to define s.

However, given a quasiclassical vertex operator algebra V^0, can we find a quantum vertex operator algebra V such that $V^0 = V/qV$?

For the example of a finitely generated commutative associative algebra one can show this directly. In general it is not clear if such a quantum vertex operator algebra even exists.

Proposition 3.5. *Let V^0 be a finitely generated commutative algebra. Then any structure s of a quasiclassical vertex operator algebra on V^0 can be quantized, i.e. there exists a quantum vertex operator V such that $V^0 = V/qV$.*

3.2. Non-degenerate vertex operator algebras. As the additional braided associativity assumption on quantum vertex operator algebras and our demands on S might seem very strong we will see that in the non-degenerate case, these actually boil down to either be trivial, respectively equivalent to ordinary associativity.

Definition 3.6. A vertex operator algebra is said to be non-degenerate if

$$Z_n = Y(z_1)(\mathrm{Id} \otimes Y(z_2)) \ldots \left(\mathrm{Id}^{\otimes n-1} \otimes Y(z_n)\right)\left(\mathrm{Id}^{\otimes n} \otimes \Omega\right) : V^n \otimes \Bbbk(z_1, \ldots, z_n) \to V((z_1)) \ldots ((z_n))$$

are injective $\forall n \in \mathbb{N}$

Proposition 3.7. *Let (V, Y, T, Ω, S) be a data satisfying the axioms of a braided vertex operator algebra except for the quantum Yang-Baxter equation and unitarity. Then we have that V automatically is a braided vertex operator algebra (i.e. the above axioms actually hold) and the quantum hexagon relation (1) is equivalent to associativity (2).*

4. Examples

In this section we give examples of quantum vertex algebras, one easier one, and one more complicated. I will discuss these deeper in the talk if time permits.

4.1. finitely generated commutative associative algebra. Let A be a finitely generated commutative associative algebra and set $T = 0$ then we have that all the defining properties boil down to $Y(z)$ being a constant polynomial (degree 1) and so is the ordinary product, which forces $\Omega = 1$. And as we seen above if we even are given a quasiclassical r-matrix we can also quantify that.

4.2. The quantum affine vertex operator algebra. For a rational, trigonometric or elliptic R-matrix we can construct the quantum function algebra $F_0(R)$ as a kind of "free topological" Hopf algebra over R. Considering the quantum universal enveloping algebra $U_0(R)$ and a $K \in \Bbbk$ we can make this into a quantum vertex operator algebra. Furthermore we get that the quasiclassical limit of $U_0(R)$ is the affine vertex operator algebra for $\mathfrak{g} = \mathrm{sl}_N$.

For R rational, this can be done on a more conceptual level called the "Coinvariant Construction"

References

[EK98] P. Etingof and D. Kazhdan, *Quantization of Lie bialgebras, V*, (August 1998), math/9808121.