1 Vertex Coalgebras

We follow the definition of [Hub08, Def. 3.3] for vertex coalgebras.

Definition 1.1. A $\mathbb{Z}$-graded **vertex coalgebra** consists of a $\mathbb{Z}$-graded vector space $V = \coprod_{k \geqslant -K} V_k$, with $K \in \mathbb{Z}_{\geqslant 0}$, a map $c : V \to \mathbb{C}$, called the **covacuum map**, and a linear operation

$$\check{Y}(z) : V \to (V \otimes V)((z^{-1})),\ \check{Y}(z)(v) = \sum_{n \in \mathbb{Z}} \Delta_n(v) z^{-n-1},$$

called the **vertex coproduct**, such that $\Delta_n(v)$ is in $(V \otimes V)_{k+n+1}$ for a homogeneous element v in V_k. These have to satisfy the following axioms for $v \in V$ and $v' \in \coprod_k (V^{\otimes 3})^*_k$

(a) (*Left counit axiom*) $(c \otimes \mathrm{Id}_V)\check{Y}(z) = \mathrm{Id}_V$.

(b) (*Cocreation axiom*) $(\mathrm{Id}_V \otimes c)\check{Y}(z)(v) \in V[[z]]$ and $(\mathrm{Id}_V \otimes c)\check{Y}(0)(v) = v$.

(c) $\check{Y}(z)\mathrm{D}^* - (\mathrm{Id}_V \otimes \mathrm{D}^*)\check{Y}(z) = \partial_z \check{Y}(z)$, where $\mathrm{D}^* = (\mathrm{Id}_V \otimes c)\Delta_{-2}$.

(d) (*Cocommutativity axiom*) There exists an $N \in \mathbb{Z}_{\geqslant 0}$, depending on v' but not on v, such that

$$(z-w)^N v' \left(\left(\mathrm{Id}_V \otimes \check{Y}(w) \right) \check{Y}(z)(v) - (T \otimes \mathrm{Id}_V) \left(\mathrm{Id}_V \otimes \check{Y}(z) \right) \check{Y}(w)(v) \right) = 0$$

as elements in $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$, where $T : V \otimes V \to V \otimes V$ is the transposition operator.

Construction 1.2. Given a vertex algebra structure on a given graded vector space V, we obtain a vertex coalgebra structure on $V^\vee = \coprod_{k \geqslant -K} V_k^*$ as follows.

We define the covacuum map to be $c : V^\vee \to \mathbb{C},\ p \mapsto p\,|0\rangle$. We assume each V_k is finite-dimensional, so that V is isomorphic as a graded vector space to $V^\vee$. In this case, we can define $\check{Y}$ by the formula

$$\check{Y}(z)(p)(A \otimes v) = p\,(Y(A,z)v)\,.$$

The vertex coalgebra axioms can then be checked using the vertex algebra axioms of $(V, |0\rangle, T, Y)$ and the defining formula above.

We will encounter this situation in later talks, where the homology of the moduli stack of an abelian category obtains a vertex algebra structure, defined by [Joy21]. The above situation then applies to give a vertex coalgebra structure to cohomology of this moduli stack, which is further studied in [Lat21].

2 Vertex Lie Algebras

2.1 Definitions and Constructions

We follow the definitions and constructions in [FB04, Ch. 16] based on [Pri99].

For any formal power series $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n$, we write $a(z)_- = \sum_{n \leqslant 0} a_n z^n$, which is called its **polar part**.

Definition 2.1. A $\mathbb{Z}$-graded **vertex Lie algebra** consists of a $\mathbb{Z}$-graded vector space L, a linear operator $D : L \to L$ of degree 1, and a linear operation

$$Y_{-}(\cdot, z) : L \to \operatorname{End}(L) \otimes z^{-1} \left[\left[z^{-1}\right]\right], \; Y_{-}(A, z) = \sum_{n \geqslant 0} A_{(n)} z^{-n-1},$$

such that for any $v \in L$ we have $A_{(n)}v = 0$ for sufficiently large n. For an element A of degree m, we further require $A_{(n)}$ to have degree $m-n-1$. These have to satisfy the following axioms:

(a) (*Translation axiom*) For any $A \in L$, we have $Y_{-}(DA, z) = \partial_z Y_{-}(A, z)$.

(b) (*Skew-symmetry axiom*) For any $A, B \in L$, we have $Y_{-}(A, z)B = \left(e^{zD} Y_{-}(B, -z)A\right)_{-}$.

(c) (*Commutator axiom*) For any $A, B \in L$, we have

$$\left[A_{(m)}, Y_{-}(B, w)\right] = \sum_{n \geqslant 0} \binom{m}{n} \left(w^{m-n} Y_{-}(A_{(n)}B, w)\right)_{-} .$$

Here, the commutator is defined by linearly extending from homogeneous elements, and for homogeneous elements as $[v, w] = vw - (-1)^{\deg(v)\deg(w)} wv$.

Note that the commutator axiom is equivalent to

$$\left[A_{(m)}, B_{(k)}\right] = \sum_{n \geqslant 0} \binom{m}{n} \left(A_{(n)}B\right)_{(m+k-n)} \tag{1}$$

for any $m, k \geqslant 0$, by collecting the individual terms.

Definition 2.2. Given a vertex algebra $(V, |0\rangle, D, Y)$ we define a vertex Lie algebra (V_{-}, D_{-}, Y_{-}), called its **polar part** as follows. We take $V_{-} = V$, $D_{-} = D$, and if $Y(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$, then we set

$$Y_{-}(A, z) := \sum_{n \geqslant 0} A_{(n)} z^{-n-1},$$

or in other words $Y_{-}(\cdot, z) = (Y(\cdot, z))_{-}$.

Remark 2.3. This indeed makes (V_{-}, D_{-}, Y_{-}) a vertex Lie algebra. The translation axiom and the skew-symmetry axiom follow from taking the polar part of the corresponding formulas for vertex algebras. Note that the translation axiom makes sense as differentiation preserves the polar part of a formal power series.

The commutator formula (1) follows from a corresponding formula for the commutators of coefficients of vertex operators in vertex algebras [FB04, (3.3.12)].

We consider Borcherds' construction of a Lie algebra associated to a vertex algebra. By construction, it suffices to consider the polar part of the vertex algebra to recover this Lie algebra.

Construction 2.4. Borcherds' Lie algebra[Bor86] is constructed from a given vertex algebra $(V, |0\rangle, D, Y)$ by taking V/DV with the bracket $[u, v] = u_{(0)}v$.

If we first start with a vertex Lie algebra (L, D, Y_-), we define a Lie algebra L/DL, with bracket $[u, v] = u_{(0)}v$. The bracket can be checked to be well-defined using the translation and skew-symmetry axioms. The anti-symmetry follows from skew-symmetry of the vertex Lie algebra, and the Jacobi identity follows from (1).

Moreover, by construction, we obtain the same Lie algebra from the polar part of a vertex algebra as from the full vertex algebra.

2.2 Local Lie Algebra

Based on Borcherds' construction above, we can another Lie algebra to a vertex Lie algebra, called its local Lie algebra.

Definition 2.5. Given a vertex Lie algebra (L, D, Y_-), we define its local Lie algebra as

$$\mathcal{L}(L) := \left(L \otimes \mathbb{C}[t^{\pm 1}]\right) / \operatorname{Im}(D \otimes 1 + \operatorname{Id} \otimes \partial_t).$$

We write $A_{[n]}$ for the class of $A \otimes t^n$ in $\mathcal{L}(L)$. $\mathcal{L}(L)$ is the span of these elements. The Lie bracket is defined by

$$\left[A_{[n]}, B_{[k]}\right] = \sum_{n \geqslant 0} \binom{m}{n} \left(A_{(n)}B\right)_{[m+k-n]}. \tag{2}$$

We define $\mathcal{L}(L)_+$ to be the Lie-subalgebra which is generated by all $A_{[n]}$ with $n \geqslant 0$.

The Lie algebra $\mathcal{L}(L)$ is equipped with a derivation D, defined by $DA_{[n]} = (DA)_{[n]}$ on the generators.

Remark 2.6. To show that the above construction yields a well-defined Lie algebra, we show that $L \otimes \mathbb{C}[t^{\pm 1}]$ becomes a vertex Lie algebra with translation operator $D \otimes 1 + \operatorname{Id} \otimes \partial_t$ and vertex operators which produce the above commutator via Borcherds' construction.

3 Vertex Enveloping Algebras

Given a vertex Lie algebra L, we will construct its associated vertex enveloping algebra. This operation is the left adjoint to taking the polar part of a vertex algebra, giving a vertex Lie algebra.

Definition 3.1. Denote the universal enveloping algebras of $\mathcal{L}(L)$ and $\mathcal{L}(L)_+$ by $\mathcal{U}(L)$ and $\mathcal{U}(L)_+$ respectively. Let $\mathbb{C}$ be the trivial one-dimensional representation of $\mathcal{L}(L)_+$ viewed as a $\mathcal{U}(L)_+$-module. Then the **vertex enveloping algebra** of the vertex Lie algebra L is

$$\mathcal{V}(L) = \mathcal{U}(L) \otimes_{\mathcal{U}(L)_+} \mathbb{C}.$$

The vacuum vector is $|0\rangle = 1 \otimes 1$.

There is a vertex algebra structure on $\mathcal{V}(L)$ and this construction is adjoint to the construction of the polar part of a vertex algebra.

Lemma 3.2. *Take the vacuum vector $|0\rangle = 1 \otimes 1$ defined above, and the translation operator defined by $D\,|0\rangle = 0$, $\left[D, A_{[n]}\right] = -nA_{[n-1]}$. The vertex operators can be uniquely extended to a vertex algebra structure on $\mathcal{V}(L)$ from the definition*

$$Y\left(A_{[-1]}\,|0\rangle, z\right) = \sum_{n \in \mathbb{Z}} A_{[n]} z^{-n-1},$$

where $A_{[n]}$ is viewed as an endomorphism of $\mathcal{V}(L)$ by its $\mathcal{U}(L)$-module structure.

Moreover, the construction of the vertex enveloping algebra $\mathcal{V}(\cdot)$ of a vertex Lie algebra is left adjoint to taking the polar part V_- of a vertex algebra V. More precisely, for any vertex Lie algebra L and vertex algebra V, we have a canonical isomorphism

$$\mathrm{Hom}_{VA}\left(\mathcal{V}(L), V\right) \cong \mathrm{Hom}_{VLA}\left(L, V_-\right).$$

Proof Sketch. The vertex algebra structure is constructed as in the examples of the Virasoro and Kac-Moody vertex algebras. By the Poincaré-Birkhoff-Witt theorem, $\mathcal{V}(L)$ has a basis of certain monomials in $A_{[n]}\,|0\rangle$, where $A \in L$ and $n < 0$. The commutators (2) of the local Lie algebra can be used to check that the fields $Y\left(A_{[-1]}\,|0\rangle, z\right)$ are mutually local. Then the reconstruction theorem gives the desired vertex algebra structure on $\mathcal{V}(L)$.

A homomorphism of vertex Lie algebras $L \to V_-$ can be extended to a homomorphism of vertex algebras $\mathcal{V}(L) \to V$ using the above basis of monomials $A^{(1)}_{[n_1]} \cdots A^{(k)}_{[n_k]}\,|0\rangle$ given by the Poincaré-Birkhoff-Witt theorem.

We identify L with a subset of $\mathcal{V}(L)$ by mapping A to $A_{[-1]} \otimes 1$. This map is an injective morphism of vertex Lie algebras [Pri99, Prop. 5.4]. So we can restrict a morphism of vertex algebras $\mathcal{V}(L) \to V$ to a morphism of vertex Lie algebras $L \to V_-$. ☐

Example 3.3. The construction of the above vertex enveloping algebra is very similar to the construction of the Virasoro and Kac-Moody vertex algebras. In both cases a completion of the local Lie algebra of the vertex Lie algebra recovers the Virasoro and Kac-Moody Lie algebra respectively. Then the vertex enveloping algebras

$$\mathcal{V}(L) = \mathcal{U}(L) \otimes_{\mathcal{U}(L)_+} \mathbb{C}$$

exactly recover the Virasoro and Kac-Moody vertex algebras Vir_0 and $V_0(\mathfrak{g})$ respectively.

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