EXAMPLES OF VERTEX ALGEBRAS

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1. COMMUTATIVE VERTEX ALGEBRAS

A simple class of vertex algebras can constructed via the data of a commutative algebra with derivation. More precisely, let V be a commutative algebra with derivation T, and set $|0\rangle = 1$. We define the state-field correspondence via

$$Y(a,z) := e^{zT}a \in End(V)[[z^{\pm 1}]]$$

Then the data $(V, |0\rangle, T, Y)$ defines a vertex algebra as defined in the first talk.

We first verify the **vacuum axioms**:

$T(1) = T(1.1) = T(1).1 + 1.T(1) = 2T(1)$ so $T(|0\rangle) = 0$. Also, $Y(1,z).b = (e^{zT}.1)b = b$ and $Y(a,z).1 = (e^{zT}a).1 \equiv a \mod z$.

We next check the **translation axiom**:

$$[T, Y(a,z)]b = T(Y(a,z)b) - Y(a,z)T(b) = T(e^{zT}b) - (e^{zT}a)T(b)$$

$$= T(ab + T(a)bz + T^2(a)b\frac{z^2}{2} + \cdots)$$
$$-aT(B) - T(a)T(b)\frac{z^2}{2} - \cdots$$

$$= (T(a) + T^2(a)z + T^3(a)\frac{z^2}{2} + \cdots)b$$

$$= \partial_z(a + T(a)z + T^2(a)\frac{z^2}{2} + \cdots)$$

$$= (\partial_z Y(a,z))b$$

Finally, we observe the **locality axiom** trivially holds since V is commutative.

Another natural example of a vertex algebra comes from infinitessimal families of endomorphisms of a space U over a punctured disc.

Theorem 1.1. Let U be a vector space and $V \subset End(U)[[z^{\pm 1}]]$ such that

(1) $1 \in V$
(2) V is ∂_z-invariant
(3) V is closed under all n-products of formal distributions.

Define $T \in End(V)$ by $T(a(w))\partial_w a(w)$, and Y to be the natural product induced by $End(U)[[z^{\pm 1}]]$. Then $(V, |0\rangle = 1, T, Y)$ is a vertex algebra.

2. Affine Kac-Moody algebras

Let $\mathfrak{a}$ be a Lie algebra with symmetric bilinear form κ satisfying

$$\kappa([x,y],z) = \kappa(x,[y,z]) = 0$$

for all $x, y, z \in \mathfrak{a}$.

Definition 2.1. The Kac-Moody affinisation of $(\mathfrak{a}, \kappa)$ is the lie algebra with underlying vector space

$$\hat{\mathfrak{a}}_\kappa := \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$$

with lie bracket

$$[xt^m, yt^m] := [x,y]t^{m+n} + m\delta_{m+n,0}\kappa(x,y)\mathbf{1}, \ \ [\mathbf{1}, \hat{\mathfrak{a}}_\kappa] = 0$$

.

Definition 2.2. An $\hat{\mathfrak{a}}_\kappa$-module M is called **smooth** if, for all $a \in \hat{\mathfrak{a}}_\kappa$, the formal power series

$$x(z) := \sum_{n\in\mathbb{Z}} (xt^n) z^{-n-1}$$

is a field on M.

Proposition 2.3. Let M be a smooth $\hat{\mathfrak{a}}_\kappa$-module such that $\mathbf{1}$ acts as the identity. Then

$$V := \langle x(z) \ : \ x \in \mathfrak{a} \rangle_M$$

has the structure of a vertex algebra. By this notation, we mean take the images of the $x(z)$ under the representation to $\mathfrak{gl}(M)$, so we are working with power series with $End(M)$ coefficients.

The proposition is an application of the following theorem:

Theorem 2.4. Let U be a vector space and $\mathcal{V}$ a subspace of fields on U such that

(1) $a(z), b(z)$ are mutually local for all $a(z), b(z) \in \mathcal{V}$.
(2) $id_U \in \mathcal{V}$
(3) $a(z)_{(n)} b(z) \in \mathcal{V}$ for all $n \in \mathbb{Z}$.

Then, setting $|0\rangle = id_U, T = \partial_z, Y(a(z), \zeta) = \sum_{n\in\mathbb{Z}} a(z)_{(n)} \zeta^{-n-1}$
gives a vertex algebra structure on $\mathcal{V}$.

Remark 2.5. There is another to think of these affine Kac-Moody algebras in terms of Cartan matrices. A Cartan matrix for a semi-simple lie algebra encodes the lie bracket, and satisfies properties such as the diagonal entries are all 2. Considering any Cartan matrix, one can ask whether a lie algebra induces it. A space with bracket can be constructed from this data, and these are precisely Kac-Moody algebras. In general, they look like central extensions of $\mathfrak{g}[t, t^{-1}]$ for $\mathfrak{g}$ a semi-simple lie algebra.

3. The reconstruction theorem

We now state a useful theorem which will allow us to construct vertex algebras, such as lattice vertex algebras without having to explicitly write down the axioms each time.

Theorem 3.1 (The reconstruction theorem)**.** Let V be a vector space and $\mathbf{1} \in V$, $D : V \to V$ a linear map and $\{a^\alpha(z)\}_\alpha$ a countable collection of fields on V such that the following properties hold

(1) $a^\alpha(z)\mathbf{1} \in V[[z]]$ for all α. (We then can set $a^\alpha := a^\alpha(0)\mathbf{1} \in V$).
(2) $D(\mathbf{1}) = 0, [D, a^\alpha(z)] = \partial_z a^\alpha(z)$ for all α.
(3) $a^\alpha(z), a^\beta(z)$ are mutually local for all α, β.
(4) V is spanned by $a^{\alpha_1}_{(n_1)} \circ \cdots a^{\alpha_m}_{(n_m)}(\mathbf{1})$ for all $m \geq 0, \alpha_1, \cdots, \alpha_m$ indices, $n_1, \cdots, n_m < 0$.

Then, there exists a unique vertex algebra $(V, |0\rangle = \mathbf{1}, T = e^{zD}, Y)$ with $a^\alpha(z) = Y(a^\alpha, z)$ for all α.

We do not prove this theorem here, but remark that the state-field correspondence is defined by

$$Y(a^{\alpha_1}_{(n_1)} \circ \cdots a^{\alpha_m}_{(n_m)}(\mathbf{1}), z) =$$

$$\frac{1}{\prod_{i=1}^m (-n_i - 1)!} : \frac{d^{-n_1-1}}{dz^{-n_1-1}} a^{\alpha_1}(z) \cdots \frac{d^{-n_m-1}}{dz^{-n_m-1}} a^{\alpha_m}(z) :$$

where here we use the normal-ordered product as defined in the first talk. This is exactly the form of the state-field correspondence given for the Virasoro vertex algebra in the last talk.

Example 3.2. Let $V = \mathbb{C}[x_1, x_2, \cdots]$ the vector space of polynomials in countably many variables. Set $\mathbf{1} = 1$ and define $D(p) = \sum_{n \geq 1} n x_{n+1} \partial_{x_n} p$. Define a field on V , $a(z) = \sum_{n \in Z} a_{(n)} z^{-n-1}$ where

$$a_{(n)} : V \to V$$

$$p \mapsto \begin{cases} x_{-n} p, \ n < 0 \\ 0, \ n = 0 \\ n \partial_{x_n} p, \ n > 0 \end{cases}$$

It is clear that V is spanned by $a_n(1), n < 0$ and it can be checked the other properites of the reconstruction theorem also hold. We therefore get a vertex algebra, called the **Heisenberg vertex algebra**.

4. Lattice Vertex algebras

Throughout this section, we let L denote a lattice and $(.)$ be a symmetric bilinear form on L. We say L is **integral** if $(.) : L \times L \to \mathbb{Z}$ and **even** if $(.) : L \times L \to 2\mathbb{Z}$. For the remainder of this section, we follow [Joy21], but a more in depth treatment using Fock spaces is discussed in [Kel17].

Definition 4.1. Define the vector space associated to a lattice L as

$$V_L := \mathbb{C}[L] \otimes Sym(t\Lambda_{\mathbb{C}}[t])$$

where $\mathbb{C}[L]$ is the group algebra of L with basis $e^\lambda, \lambda \in L$ and $Sym(t\Lambda_{\mathbb{C}}[t])$ consists of polynomials in variables $t^a \lambda, a \in \mathbb{N}, \lambda \in L$.

In particular, V_L is spanned by elements of the form

$$e^{\lambda_0} \otimes (t^{a_1}\lambda_1) \otimes \cdots (t^{a_m}\lambda_m)$$

Example 4.2. Let $L = \mathbb{Z} \subset \mathbb{R}$. Then $V_L \cong \mathbb{C}[q^{\pm 1}, x_{(i,j)} \ : \ (i,j) \in \mathbb{N} \times \mathbb{Z}]$

It turns out that V_L admits the structure of a vertex operator algebra under suitable conditions.

Theorem 4.3. Associated to any even lattice L (with non-degenerate symmetric bilinear form $(.)$), the space V_L admits the structure of a vertex algebra. Moreover, if $(.)$ is positive definite this can be enhanced to the structure of a vertex operator algebra.

Again, we can show this by the reconstruction theorem. We saw that V_L is spanned by elements of the form

$$e^{\lambda_0} \otimes (t^{a_1}\lambda_1) \otimes \cdots \otimes (t^{a_n}\lambda_n)$$

for $\lambda_1, \cdots, \lambda_n \in L, a_1, \cdots, a_n > 0$. Define a $\mathbb{C}$-derivation $D : V_L \to V_L$ by

$$D(e^\lambda \otimes 1) = e^\lambda \otimes (t\lambda)$$
$$D(e^0 \otimes (t^a\lambda)) = ae^0 \otimes (t^{a+1}\lambda)$$

Furthermore, for $\mu \in L$ and $n \in \mathbb{Z}$ define

$$\mu_n : V_L \to V_L$$

by

(1) If $n > 0$, $\mu_n(e^\lambda \otimes 1) = 0$.
(2) If $n = 0$, $\mu_0(e^\lambda \otimes p) = (\mu, \lambda)e^\lambda \otimes p$
(3) If $n < 0$, μ_n is multiplication by $e^0 \otimes (t^{-n}\mu)$.

We now set $\mu(z) := \sum_{n\in\mathbb{Z}} \mu_n z^{-n-1}$. Additionally, define $\tilde{\mu}(z) : V_L \to V_L[[z]][z^{-1}]$ by

$$\tilde{\mu}(z) = \epsilon_{\lambda,\mu} z^{(\lambda,\mu)}(e^\mu \otimes 1). \exp[-\sum_{n<0} \frac{z^{-n}}{n}\mu_n] \circ \exp[-\sum_{n>0} \frac{z^{-n}}{n}\mu_n](e^\lambda \otimes p)$$

The $\epsilon_{\lambda,\mu}$ appearing here are a choice of signs for each pair $\lambda, \mu \in L$ satisfying the following three conditions:

(1) $\epsilon_{\lambda,0} = \epsilon_{0,\mu} = 1$
(2) $\epsilon_{\lambda,\mu}.\epsilon_{\mu,\lambda} = (-1)^{(\lambda,\mu)+(\lambda,\lambda)(\mu,\mu)}$
(3) $\epsilon_{\lambda,\mu} = \epsilon_{\lambda+\mu,\nu} = \epsilon_{\lambda,\mu+\nu}.\epsilon_{\mu,\nu}$

Then, if we pick a $\mathbb{Z}$-basis $\mu_1, \cdots, \mu_d \in L$ of L, we see that

$$\{\mu_1(z), \tilde{\mu}_1(z), \cdots, \mu_d(z), \tilde{\mu}_d(z)\}$$

is a collection of fields on the space V_L that satisfy the conditions of the reconstruction theorem. We therefore get a vertex algebra called a **lattice vertex algebra**.

If $(.)$ is non-degenerate, the central charge of the associated vertex operator algebra is d, the rank of L, and the conformal vector is

$$\omega = \frac{1}{2} \sum_{i,j=1}^{d} A_{i,j} e^0 \otimes (t\mu_i) \otimes (t\mu_j)$$

where (A_{ij}) is the inverse matrix to $((\mu_i, \mu_j))_{i,j}$.

It turns out that if $(.)$ is non-degenerate, then the subspace $W \subset V_L$ spanned by vectors of the form $e^0 \otimes p$ is a sub-vertex algebra isomorphic to $\mathfrak{H}^{\otimes d}$, for $\mathfrak{H}$ the Heisenberg vertex algebra. For example, as in Example 4.2, the sub-algebra $W = \mathbb{C}[x_{(i,j)} \,:\, (i,j) \in \mathbb{N} \times \mathbb{Z}]$ admits a vertex algebra structure making it isomorphic to the Heisenberg vertex algebra.

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