COHOMOLOGICAL HALL ALGEBRAS AND VERTEX ALGEBRAS

1. Various structures from moduli stack

Let X be a stack parametrizing objects in an abelian category $\mathcal{A}$ of dimension at most one, e.g.

$$\mathcal{A} = \mathrm{Rep}(Q) \text{ or } \mathrm{Coh}(C)$$

where Q is a quiver without relations and C is a smooth projective curve. There are two algebraic structures on $H^*(X)$; cohomological Hall algebra and braided vertex coalgebra. The main result of [1] is a compatibility between the two structures, hence making $H^*(X)$ a braided vertex bialgebra.

A set of connected components of the stack, denoted by $\pi_0(X)$, is given by

$$\pi_0(X) = \begin{cases} \mathbb{N}^{|Q|} & \text{if } \mathcal{A} = \mathrm{Rep}(Q), \\ \big\{(r,d) \,\big|\, (r \in \mathbb{Z}_{>0},\ d \in \mathbb{Z}) \text{ or } (r = 0,\ d \in \mathbb{Z}_{\geq 0})\big\} & \text{if } \mathcal{A} = \mathrm{Coh}(C). \end{cases}$$

For each $a \in \pi_0(X)$, we denote the corresponding component by X_a. In particular, $X_0 = \mathrm{pt}$.

1.1. **Cohomological Hall algebra.** Consider the universal extension diagram

$$\begin{array}{ccc} & \mathrm{Ext} & \\ {}^{p}\swarrow & & \searrow^{q} \\ X \times X & & X \end{array} \tag{1}$$

where Ext is a stack parametrizing the short exact sequences $0 \to F_1 \to F \to F_2 \to 0$. Since $\mathcal{A}$ is an abelian category of dimension at most one, both X and Ext are smooth Artin stacks. Furthermore, q is represented by a relative Quot scheme, hence is proper. We define a CoHA product as a composition

$$m := q_* \circ p^* : H^*(X) \otimes H^*(X) \to H^*(\mathrm{Ext}) \to H^{*-2\dim(q)}(X).$$

Together with a unit $1 \in H^*(X_0)$, we obtain an algebra $(H^*(X), m, 1)$.

1.2. **Braided vertex coalgebra.** Consider the following geometric structures on X:

(i) Zero object $0 : \mathrm{pt} \to X$,
(ii) Direct sum $\oplus : X \times X \to X$,
(iii) Scalar multiplication $\rho : B\mathbb{G}_m \times X \to X$,
(iv) Universal extension complex $\theta \in \mathrm{Perf}(X \times X)$.

The first two structures induce a counit and a coproduct

$$\epsilon : H^*(X) \to \mathbb{Q}\,, \quad \oplus^* : H^*(X) \to H^*(X) \otimes H^*(X),$$

forming a coalgebra $\big(H^*(X), \oplus^*, \epsilon\big)$.[1] On the other hand, the scalar multiplication map induces

$$\rho^* : H^*(X) \to H^*(X)[z]$$

where $z = c_1(\mathcal{L}) \in H^2(B\mathbb{G}_m)$ is given by the universal line bundle. By taking a coefficient of z^1 from ρ^*, we obtain a translation

$$T : H^*(X) \to H^*(X).$$

In fact, we can show that $\rho^* = e^{zT}$, hence recovering the full operator.

From the structures we have encountered above, we obtain a holomorphic vertex coalgebra:

$$\Delta_z^{\mathrm{hol}} : H^*(X) \to H^*(X) \otimes H^*(X)[z], \quad v \mapsto \rho_1^* \circ \oplus^*(v).$$

To obtain a more interesting structure than a holomorphic one, we use the last ingredient.

[1]Typically, this coalgebra structure does not form a bialgebra with CoHA. The main result fixes this issue.

The universal extension complex θ is defined by the universal object of the moduli stack. In the case of $\mathcal{A} = \mathrm{Coh}(C)$, we have

$$\theta = \mathrm{R}\pi_{12*}\, \mathrm{R}\mathcal{H}\mathrm{om}(\pi_{13}^*\mathcal{F}, \pi_{23}^*\mathcal{F}) \in \mathrm{Perf}(X \times X)\,,$$

where $\mathcal{F}$ is the universal sheaf on $X \times C$ and π_{ij}'s are the projections from $X \times X \times C$. The case of $\mathcal{A} = \mathrm{Rep}(Q)$ can be treated similarly. Crucially for us, θ has nontrivial ρ-weights, i.e.,

$$\rho_1^*(\theta) = \mathcal{L}^\vee \boxtimes \theta, \quad \rho_2^*(\theta) = \mathcal{L} \boxtimes \theta.$$

Even though the Euler class $e(\theta) \in H^*(X \times X)$ does not make sense in general, its pullback under ρ_1 can be formally defined as follows:

$$\rho_1^* e(\theta) := \prod_{i \in I}(-z + x_i) = (-z)^{\mathrm{rk}(\theta)} \cdot \sum_{k \geq 0} (-z)^{-k} c_k(\theta) \in H^*(X \times X)(\!(z^{-1})\!)$$

where $\{x_i\}_{i \in I}$ is a set of Chern roots of θ. We twist the holomorphic vertex coalgebra structure using $\rho_1^* e(\theta)$ and define

$$\Delta_z : H^*(X) \to H^*(X) \otimes H^*(X)(\!(z^{-1})\!), \quad v \mapsto \rho_1^* e(\theta) \cdot \rho_1^* \circ \oplus^*(v)$$

where $\cdot$ denotes the cup product of $H^*(X \times X)$. The resulting structure $(H^*(X), \epsilon, T, \Delta_z)$ is a <u>non-local vertex coalgebra</u>.

The failure of locality can be beautifully explained by an $R(z)$-morphism

$$R(z) := \rho_1^* e(\theta)/\rho_1^* e(\sigma^* \theta^\vee) \cdot : H^*(X) \otimes H^*(X) \to H^*(X) \otimes H^*(X)(\!(z^{-1})\!)$$

which satisfies the Yang-Baxter equation

$$R_{12}(z) \circ R_{13}(z+w) \circ R_{23}(w) = R_{23}(w) \circ R_{13}(z+w) \circ R_{12}(z). \tag{2}$$

The resulting structure $(H^*(X), \epsilon, T, \Delta_z, R(z))$, or its variants using orientation data, is what's called a <u>braided vertex coalgebra</u> in Section 2.

1.3. **Orientations.**

Braided orientation data are

$$\delta^{(1)}, \delta^{(2)} : \pi_0(X \times X) \to \{\pm 1\}$$

which are required to satisfy various conditions according to the additive structure of $\pi_0(X)$. They can be used to define a variant of $(H^*(X), \epsilon, T, \Delta_z, R(z))$ by replacing the operators with $\eta_{\delta^{(1)}} \cdot \Delta_z$ and $\delta^{(2)} \cdot R(z)$ where $\eta_{\delta^{(1)}}$ is another sign defined as $(-1)^{i \cdot \mathrm{rk}\theta_{b,b}} \cdot \delta^{(1)}_{a,b}$ on $H^i(X_a) \otimes H^*(X_b)$.

1.4. **Preview of the connection between the two structures.**

Before going into any detail, we point out one important connection between the two constructions we have seen. That is, the universal extension complex $\theta \in \mathrm{Perf}(X \times X)$ is closely related to the diagram (1) via the identity

$$T_q = p^* \theta \in \mathrm{Perf}(\,\mathrm{Ext}\,).$$

This is crucially used in the proof of the main result.

2. Braided vertex bialgebra

This section introduces a rather sophisticated algebraic object, called braided vertex bialgebra. We first look at its non-vertex counterpart, namely a quasitriangular bialgebra.

2.1. **Quasitriangular bialgebra.** A quasitriangular bialgebra consists of

(i) an algebra $(A, m, 1)$,
(ii) a coalgebra (A, Δ, ϵ),
(iii) an R-morphism[2] $R : A \otimes A \to A \otimes A$,

which are required to satisfy the following axioms:

(A1) Almost cocommutativity: $\Delta^{\mathrm{op}} = R \circ \Delta$,
(A2) Hexagon identities: $(\Delta\otimes\mathrm{id})\circ R = R_{13}\circ R_{23}\circ(\Delta\otimes\mathrm{id})$ and $(\mathrm{id}\otimes\Delta)\circ R = R_{13}\circ R_{12}\circ(\mathrm{id}\otimes\Delta)$,
(A3) Bialgebra axiom[3]: we have $\Delta(a)\cdot\Delta(b) = \Delta(a\cdot b)$, i.e., the following diagram commutes

$$\begin{array}{ccccc} A\otimes A & \xrightarrow{\ m\ } & A & \xrightarrow{\ \Delta\ } & A\otimes A \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \uparrow{\scriptstyle m\otimes m} \\ A\otimes A\otimes A\otimes A & & \xrightarrow[\mathrm{id}\otimes\sigma\otimes\mathrm{id}]{} & & A\otimes A\otimes A\otimes A. \end{array}$$

Recall that $\Delta^{\mathrm{op}} := \sigma\circ\Delta$. So (A1) measures the failure of cocommutativity with an R-morphism.

Example 1. Quantum group $U_q(\mathfrak{g})$ is an example of quasitriangular bialgebra. Indeed, quantum groups often fail to be cocommutative which is then explained by an R-matrix.

2.2. **Braided vertex bialgebra.** A braided vertex coalgebra consists of

(i) an algebra $(V, m, 1)$,
(ii) a non-local vertex coalgebra $(V, \epsilon, T, \Delta_z)$,
(iii) an $R(z)$-morphism $R(z) : V\otimes V \to V\otimes V(\!(z^{-1})\!)$,

satisfying the vertex analogue of the previous axioms (B1), (B2), (B3').

We first explain what we mean by a non-local vertex coalgebra. Recall that a vertex coalgebra consists of (with co- in front of everything)

(i) a vector space V,
(ii) a vacuum: $\epsilon : V \to \mathbb{Q}$,
(iii) a translation: $T : V \to V$,
(iv) a state-field correspondence: $\Delta_z : V \to V\otimes V(\!(z^{-1})\!)$,

satisfying the co-vacuum axiom, co-translation axiom and weak cocommutativity (= locality). Definition of a non-local vertex coalgebra is obtained by replacing the weak cocommutativity with a weak coassociativity.[4] The usual weak associativity of vertex algebra reads

$$(z-w)^n\Big[Y(Y(\alpha,z)\beta,-w)\gamma - Y(\alpha,z-w)Y(\beta,-w)\gamma\Big] = 0 \in V[\![z^{\pm},w^{\pm}]\!].$$

By appropriately dualizing it, weak coassociativity is the commutativity of the diagram below

$$\begin{array}{ccccc} & & V\otimes V(\!(w^{-1})\!) & & \\ & \nearrow^{\Delta_{-w}} & & \searrow^{\Delta_z\otimes\mathrm{id}} & \\ V & & & & V\otimes V\otimes V[\![z^{\pm},w^{\pm}]\!] \\ & \searrow_{\Delta_{z-w}} & & \nearrow_{\mathrm{id}\otimes\Delta_{-w}} & \\ & & V\otimes V(\!((z-w)^{-1})\!) & & \end{array}$$

after multiplying by sufficiently large $(z-w)^n$ where n depends on the source.

Now we explain the axioms (B1), (B2) and (B3'). First of all, we define the analogue of the opposite comultiplication in the vertex setting: (the right hand side is related to skew symmetry)

$$\Delta_z^{\mathrm{op}} := \sigma\circ\Delta_{-z}\circ e^{zT}.$$

This is a correct analogue because indeed $\Delta_z^{\mathrm{op}} = \Delta_z$ implies the weak cocommutativity (= locality).

[2]In this note, we do not view R-matrix as an actual element in $A\otimes A$. So axioms below are a bit different from what is in the literature.
[3]We omit various bialgebra axioms related to (co)units.
[4]This is a strictly weaker axiom. Recall that any vertex algebra is weakly associative.

(B1) Almost cocommutativity: $\Delta_z^{\mathrm{op}} = R(z) \circ \Delta_z$,

(B2) Hexagon identities:

$$(\Delta_z \otimes \mathrm{id})R(w) = R_{13}(z+w)R_{23}(w)(\Delta_z \otimes \mathrm{id}), \quad (\mathrm{id} \otimes \Delta_z)R(w) = R_{13}(w)R_{12}(w-z)(\mathrm{id} \otimes \Delta_z),$$

(B3') $R(z)$-twisted bialgebra axiom: we have $\Delta_z(a) \cdot_{R(z)} \Delta_z(b) = \Delta_z(a \cdot b)$, i.e.,

$$\begin{array}{ccccc} V \otimes V & \xrightarrow{m} & V & \xrightarrow{\Delta_z} & V \otimes V(\!(z^{-1})\!) \\ \downarrow{\scriptstyle \Delta_z \otimes \Delta_z} & & & & \uparrow{\scriptstyle m \otimes m} \\ V \otimes V \otimes V \otimes V(\!(z^{-1})\!) & & \xrightarrow[\mathrm{id} \otimes (\sigma \circ R(z)) \otimes \mathrm{id}]{} & & V \otimes V \otimes V \otimes V(\!(z^{-1})\!). \end{array} \tag{3}$$

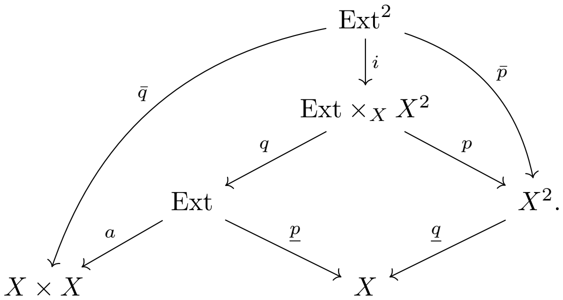
3. Compatibility between CoHA and vertex algebra

Theorem 2 (Latyntsev [1]). *The cohomology group $H^*(X, \mathbb{Q})$ is naturally a braided vertex bialgebra according to the geometrically described operators in Section 1.*

We only sketch the main idea of the proof. Recall that the formulas of the braided vertex coalgebra structure is written in terms of the universal extension complex θ. To prove the commutativity of the diagram (3), we need to understand the CoHA product also in terms of θ. Observe that the pullback

$$\oplus^* : H^*(X) \to H^*(X^2)$$

is injective because we have a zero object. Therefore it suffices to understand the CoHA product after pulling back to X^2. Key diagram is the following:



Here $\mathrm{Ext} \times_X X^2$ parametrizes exact sequences with a splitting middle term

$$0 \to F_1 \to (F^\ell \oplus F^r) \to F_2 \to 0.$$

This stack now admits a torus action by scaling one of the middle factors, say the left one. The torus fixed substack parametrizes splitting exact sequences

$$0 \to F_1^\ell \oplus F_1^r \to F^\ell \oplus F^r \to F_2^\ell \oplus F_2^r \to 0.$$

Note that this is simply Ext^2 in the above diagram. By the abelian localization technique developed in [1], we have

$$\underline{q}^* \underline{p}_* a^* = p_* q^* a^* = \bar{p}_* \frac{\bar{q}^*(-)}{e(N_i)}.$$

On the other hand, we have the K-theory identity

$$N_i = i^* T_{\mathrm{Ext} \times_X X^2} - T_{\mathrm{Ext}^2} = i^* q^* T_{\underline{p}} - T_{\bar{p}} = \bar{q}^* \theta - \left(q^* \theta \boxplus q^* \theta\right)$$

by the observation in subsection 1.4. These are the main geometric ingredients for the proof. We refer the algebraic part of the proof and details to the original reference [1].

References

[1] Alexei Latyntsev. Cohomological hall algebras and vertex algebras, 2021.