1 BPS Lie algebra talk handout

1.1 Introduction

As we have seen in the previous talk, the critical CoHA has a localised coproduct. In fact the idea of introducing the coproduct comes from thinking of the CoHA as a sort of universal enveloping algebra of a Lie algebra or quantum group. In this talk we are concerned with explaining one part of this analogy, namely the PBW theorem. Let us start with recalling the story for Lie algebras. Let $\mathfrak{g}$ be a Lie algebra, then the universal Lie algebra $U(\mathfrak{g}) = T(\mathfrak{g})/(a \otimes b - b \otimes a - [a, b])$ is a filtered algebra. Then we have the following classical theorem

Theorem 1.1.1 (PBW theorem)**.** *There is an algebra isomorphism* $\mathrm{Sym}(\mathfrak{g}) \to \mathrm{Gr}\, U(\mathfrak{g})$ *and a vector space isomorphism* $\mathrm{Sym}(\mathfrak{g}) \to U(\mathfrak{g})$.

Since the CoHA is meant to be some sort of enveloping algebra we can expect some sort of theorem like this. We already have a coproduct so we should have the following ingredients

1. a filtration on the CoHA: this will be given by the so called perverse filtration $\mathfrak{P}$ 1.3.5.
2. a Lie algebra: this will be the BPS Lie algebra $\mathfrak{g}_Q$ 1.5.6 .

With these ingredients we can try to formulate a PBW type theorem. However, this is more an analogy since in fact the universal enveloping algebra of the BPS Lie algebra will not be equal to the CoHA and the PBW theorem will instead be a **Yangian** type PBW theorem.

Theorem 1.1.2 (Davison-Meinhardt 1.5.3)**.** *Let Q be a symmetric quiver with some potential W. We have isomorphisms*

$$A_{Q,W} \cong \mathrm{Sym}(\mathrm{BPS} \otimes \mathrm{H}(\mathrm{B}\,\mathbb{C}^*)_{vir})$$
$$\mathrm{Gr}_{\mathfrak{P}}(A_{W,Q}) \cong \mathrm{Sym}(\mathrm{BPS} \otimes \mathrm{H}(\mathrm{B}\,\mathbb{C}^*)_{vir})$$

where the first is an isomorphism of graded vector spaces and the second an isomorphism of algebras.

The goal is to define the objects in this theorem and do some examples. In particular, the BPS Lie algebra can be related to Kac-Moody Lie algebras associated to the quiver Q as we will see in section 1.6.

Convention 1.1.3. To simplify things we will assume that our quiver Q is symmetric and we will state all results without mention of stability conditions or Serre subcategories. Lastly, we have ignored certain sign issues in the statements of the PBW theorems but we indicate where they need to be changed.

Notation 1.1.4. We will denote by $A_{W,Q}$ the absolute critical CoHA for a quiver with potential W and $RA_{W,Q}$ for the relative version. We denote by $\mathbb{Q}_X$ the constant sheaf on some space X. Also we denote by $\mathbb{Q}_{X,vir} = \mathbb{Q}_X[\dim X]$. We set $\mathrm{H}(\mathrm{B}\,\mathbb{C}^*)_{vir} = \mathrm{H}(\mathrm{B}\,\mathbb{C}^*, \mathbb{Q}[-1])$. For quivers Q we denote the set of vertices by Q_0 and arrows by Q_1. We will usually write moduli stacks using $\mathfrak{M}$ and coarse moduli spaces as $\mathcal{M}$.

1.2 Mixed hodge modules and decomposition theorem

Let us start with mentioning some things about mixed hodge modules. While they are a big part of the theory, we will not mention them so much to decrease the amount of technical details.

Definition 1.2.1 (Definition of mhm)**.** Let X be an algebraic complex manifold $MHM(X)$ is a full sub-tensor category of the category of triples

1. a perverse sheaf L of $\mathbb{Q}$ vector spaces
2. regular holonomic module M such that $DR(M) = L \otimes_{\mathbb{Q}} \mathbb{C}$
3. a good filtration on M

Informally we will think of Mixed Hodge Modules as perverse sheaves equipped with a weight filtration. In particular there is a faithful forgetful map $\mathrm{MHM}(X) \to \mathrm{Perv}(X)$.

Theorem 1.2.2 (Decomposition theorem). *Let $p : X \to Y$ be a projective morphism between algebraic varieties and X smooth then we have*

$$p_*\mathbb{Q}_X[\dim X] = \bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}(p_*\mathbb{Q}_X[\dim X])[-i]$$

This can be generalised to non-smooth X by taking the so called intersection complex IC_X instead of the constant sheaf.

Theorem 1.2.3 (Pure MHMs). *Category of pure MHMs is semisimple and a projective map $p : X \to Y$ preserves pure objects.*

We can use the fact that proper maps behave well with vanishing cycles to deduce the following crucial result.

Theorem 1.2.4 (Vanishing cycles decomposition). *Let $p : X \to Y$ be a proper map. Then we have a non-canonical isomorphism*

$$p_*(\phi_{fp}\mathcal{F}) \cong \phi_f(p_*\mathcal{F}) \cong \bigoplus_{n\in\mathbb{Z}} {}^p\mathcal{H}^n(\phi_f p_*\mathcal{F})[-n].$$

1.3 Perverse filtration and relative CoHA

Let $p : X \to Y$ be a projective morphism. Then if $\mathcal{F}$ is a pure mixed hodge module we have that $p_*\mathcal{F} = \bigoplus \mathcal{H}^i(p_*\mathcal{F})[-i]$. We can write the natural morphism $\tau^{\leq k}p_*\mathcal{F} \to p_*\mathcal{F}$ as

$$\beta : \bigoplus_{i\leq k} {}^p\mathcal{H}^i(p_*\mathcal{F})[-i] \to \bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(p_*\mathcal{F})[-i]$$

the splitting allows us to define an left inverse. Pushing forward to the point we get

$$\beta : \mathrm{H}(Y, \tau^{\leq k}p_*\mathcal{F}) \to \mathrm{H}(X, \mathcal{F})$$

so we still have a left inverse α.

Definition 1.3.1 (General perverse filtration). We define $\mathfrak{P}_k(H(X, \mathcal{F})) = H(Y, \tau^{\leq k}p_*\mathcal{F})$.

We want to use this theory to define a perverse filtration on the CoHA. Let us recall the set up for quivers. Let Q be a symmetric quiver. This means there are as many arrows coming out of every vertex as there are going out. Let $\mathbf{d}$ be a dimension vector. We want to use the decomposition theorem to pushforward along the affinization/semisimplification map $\mathrm{JH} : \mathfrak{M}_Q \to \mathcal{M}_Q$. However, this map is not proper so we cannot just apply the theorem. However, it turns out that it is approximated by proper maps, in the following sense.

Definition 1.3.2 (Approximation by proper maps). A morphism $p : X \to Y$ from a finite type stack X to a scheme is APM if for all $N \geq 1$ we have smooth morphisms $q_N : X_N \to X$ such that

1. $p \circ q_N$ is projective

2. for all $x \in X$ the reduced cohomology $\widetilde{H}(q_N^{-1}(x))$ is concentrated in cohomological degrees $\geq N$.

Theorem 1.3.3 (Section 4 in [4])**.** $\mathrm{JH} : \mathfrak{M}_Q \to \mathcal{M}_Q$ *is approximated by proper maps.*

The punchline is that morphisms that are approximated by proper maps also satisfy the decomposition theorem. Therefore, $\mathrm{JH}_* \, \phi_{Tr(W)} \mathbb{Q}_{vir}$ splits and we can use the technology above.

Definition 1.3.4 (Relative CoHA)**.** Define $RA_{Q,W} = \mathrm{JH}_* \, \phi_{Tr(W)} \mathbb{Q}_{\mathfrak{M}_Q,vir}$ the relative CoHA. By using similar pull push diagrams one can give this the structure of an algebra object in $D^b_c(\mathcal{M}_Q)$. For details see [4] section 5.1.

It turns out that this object is very useful for studying the absolute CoHA. The two are related by the following maps

$$\begin{array}{ccc} \mathfrak{M}_Q & \xrightarrow{\mathrm{JH}} & \mathcal{M}_Q \\ & \searrow^{\mathrm{Dim}} & \downarrow{\scriptstyle \dim} \\ & & \mathbb{N}^{Q_0} \end{array}$$

$$\mathrm{Dim}_* \phi_{Tr(W)} \mathbb{Q}_{\mathfrak{M},vir} \cong A_{W,Q} \cong \dim_* RA_{Q,W}$$

After all the setup we can finally define the perverse filtration on the absolute CoHA, using the intermediate object $RA_{Q,W}$. We use the fact that JH is approximated by proper maps so the direct image splits as well as compatibility of vanishing cyles and APM maps.

Definition 1.3.5 (Perverse filtration on CoHA)**.** We define the perverse filtration $\mathfrak{P}_k A_{Q,W} \coloneqq \mathfrak{P}_k(\mathrm{H}(\mathfrak{M}_Q, \phi_{Tr(W)} \mathbb{Q}_{\mathfrak{M}_Q,vir})) = \mathrm{H}(\mathcal{M}_Q, \tau^{\leq k} \, \mathrm{JH}_* \, \phi_{Tr(W)} \mathbb{Q}_{\mathfrak{M}_Q,vir})$. Furthermore, the CoHA products preserve the filtration, giving a filtered algebra.

1.4 Cohomological integrality

Before we can state our first version of the PBW theorem we need to define a monoidal structure on sheaves $\mathcal{F} \in D^b_c(\mathcal{M}_Q)$. For any two dimension vectors d_1, d_2 we have a map $\oplus : \mathcal{M}_{Q,d_1} \times \mathcal{M}_{Q,d_2} \to \mathcal{M}_{Q,d_1+d_2}$ by taking direct sum of representations. Then we define for $\mathcal{F} \in D^b_c(\mathcal{M}_{d_1})$ and $\mathcal{G} \in D^b_c(\mathcal{M}_{d_2})$ the tensor product

$$\mathcal{F} \otimes \mathcal{G} = \oplus_*(p_1 \mathcal{F} \otimes p_2 \mathcal{G})$$

where p_i are the projection maps to $\mathcal{M}_{Q,d_i}$. We can also calculate symmetric powers, which will appear in the following theorem, by considering S_n invariants.

Definition 1.4.1 (BPS sheaves)**.** Define

$$\mathcal{BPS}_d \coloneqq \begin{cases} \phi_{Tr(W)} IC_{\mathcal{M}_d} \text{ if } \mathcal{M}_d \neq \emptyset \\ 0 \text{ otherwise} \end{cases}$$

Upon taking cohomology we denote $\mathrm{BPS}_d = \mathrm{H}(\mathcal{M}_d, \mathcal{BPS}_d)$. Finally $\mathcal{BPS} = \bigoplus_{d \in \mathbb{N}^{Q_0}} \mathcal{BPS}_d$ and $\mathrm{BPS} = \bigoplus_{d \in \mathbb{N}^{Q_0}} \mathrm{BPS}_d$.

Here IC is the intersection complex, which is a well behaved perverse sheaf on $\mathcal{M}_d$.

Theorem 1.4.2 (Cohomological Integrality Davison-Meinhardt Theorem A in [4])**.** *Let Q be a symmetric quiver with potential. We have isomorphism in $D^b_c(\mathcal{M}_Q)$ and $D^b_c(\mathbb{N}^{Q_0})$ respectively*

$$\mathrm{JH}_* \, \phi_{Tr(W)} \mathbb{Q}_{\mathfrak{M}_Q,vir} \cong \mathrm{Sym}(\mathrm{H}(\mathrm{B}\,\mathbb{C}^*)_{vir} \otimes \mathcal{BPS})$$
$$A_{W,Q} \cong \mathrm{Sym}(\mathrm{H}(\mathrm{B}\,\mathbb{C}^*)_{vir} \otimes \mathrm{BPS}).$$

Furthermore, there is a canonical split inclusion $\gamma : \mathrm{H}(\mathrm{B}\,\mathbb{G}_m)_{vir} \otimes \mathcal{BPS} \to RA_{W,Q}$.

The proof starts by proving the statement in the $W = 0$ case and also uses purity. This theorem is a lift from the motivic level to the cohomological level of the integrality theorem of Meinhardt-Reineke [6]. This theorem is already a sort of PBW theorem, but we can say more about the exact isomorphism as well as relate it to associated graded of the perverse filtration. However, we can immediately see that the perverse filtration starts in perverse degree 1. This follows since BPS sheaves are in perverse degree 0, namely they are perverse sheaves. Since we tensor by $\mathrm{H}(\mathrm{B}\,\mathbb{C}^*)_{vir}$, which is equal to $\mathbb{C}[u][-1]$ we shift the perverse degrees by 1. This means that $\mathfrak{P}_0 = 0$. In some examples it is possible to explicitly calculate these sheaves. Let us explain one such example, but first let's state a crucial lemma.

Lemma 1.4.3 (Support lemma)**.** *Let Q be a quiver and $\widetilde{Q}$ be the tripled quiver with potential W. Let $x \in \mathcal{M}_{\widetilde{Q}}$ corresponding to a semisimple representation be in the support of $\mathcal{BPS}_d$. Then the matrices $\rho(\omega_i)$ corresponding to the action of the loop ω_i for $i \in Q_0$ have a unique generalised eigenvalue λ.*

Example 1.4.4 (Tripled Jordan quiver)**.** Here we will sketch the example following [3] section 5. Let $\widetilde{Q}_{Jor}$ be the quiver with one vertex and three loops x, y, z considered with the potential $W = x[y, z]$. Then we have that $\mathfrak{M}_{Q.d} = [\mathrm{Mat}_d^3 / GL_d]$. Now even though $\phi_{Tr(W)}\mathbb{Q}_{\mathfrak{M}_{Q,d},vir}$ is defined on $\mathfrak{M}_{Q,d}$ it will be supported on $\mathfrak{M}_{R,d} = \mathrm{crit}(Tr(W)) = [C_3(\mathrm{Mat}_d)/GL_d]$. Here $R = k[x, y, z]$ is the Jacobi algebra of our cover with potential and $C_3(\mathrm{Mat}_d)$ is the space of 3 pairwise commuting matrices. In other words, the vanishing cycles is supported on the closed subsstack of representations of the Jacobi algebra. Similarly the pushforward $\mathrm{JH}_*\,\phi_{Tr(W)}\mathbb{Q}_{vir}$ will be supported on the coarse moduli space $\mathcal{M}_{R,d} = \mathrm{Sym}^d(\mathbb{A}^3)$. This follows since the only simple $k[x, y, z]$ modules are one dimensional and the coarse moduli space is parametrising semisimple d dimensional modules. In this particular case we can use the support lemma 3 times to conclude that $\mathcal{BPS}_{\widetilde{Q},d}$ is supported on the image of the map

$$\Delta^d : \mathbb{A}^3 \to \mathcal{M}_{\widetilde{Q}_{Jor},d}$$
$$(z_1, z_2, z_3) \mapsto (z_1 I_d, z_2 I_d, z_3 I_d)$$

Furthermore, we can prove that the BPS sheaf is constant on its support, so we can conlude

$$\mathcal{BPS}_{\widetilde{Q}_{Jor},d} = \Delta^d_* \mathbb{Q}[3]$$

Let's now unpack the formula on the right hand side of the cohomological integrality theorem. The entire BPS sheaf $\mathcal{BPS} = \bigoplus_d \mathcal{BPS}_d$ lives on the disjoint union of stacks $\coprod_d \mathfrak{M}_{R,d}$ and there are 3 gradings that appear.

1. grading by the dimension vector d
2. cohomological grading, shifted by the $H(\mathrm{B}\,\mathbb{C}^*)$
3. grading by the power of Sym

We will mostly ignore the third grading but the first two are relevant as we have already seen. We can then figure out which part lives on each dimension grading

1. in dimension 1 we have $\mathcal{BPS}_1 = \Delta^1_* \mathbb{Q}_{\mathbb{A}^3}[3] = id_* \mathbb{Q}_{\mathbb{A}^3}[3] = \mathbb{Q}_{\mathbb{A}^3}[3]$
2. in dimension 2 we have $\mathcal{BPS}_2 \oplus \mathrm{Sym}^2(\mathcal{BPS}_1)$
3. in dimension 3 we have $\mathcal{BPS}_3 \oplus \mathrm{Sym}(\mathcal{BPS}_2, \mathcal{BPS}_1) \oplus \mathrm{Sym}^3(\mathcal{BPS}_1)$

If we continue like this, in dimension n there will be a term for each partition of n. One can then calculate the various sheaves more explicitly by using techniques in perverse sheaves such as IC sheaves.

1.5 PBW theorems

First lets state some important properties of the associated graded algebra to the perverse filtration that are used in the proof of PBW.

Proposition 1.5.1 (Proposition 6.7 in [4])**.** *The localised bialgebra structure on* $A_{W,Q}$ *induces a Hopf algebra structure on* $\mathrm{Gr}_{\mathfrak{P}} \mathrm{H}\, A_W$*. Furthermore,* $\mathrm{H}(\mathrm{B}\,\mathbb{C}^*)_{vir} \otimes \mathrm{BPS}$ *is a primitive subspace in* $\mathrm{Gr}_{\mathfrak{P}} A_{W,Q}$*.*

Finally, we will need a classical theorem on Hopf algebras.

Theorem 1.5.2 (Milnor-Moore)**.** *Let* A *be a graded, connected, cocommutative Hopf algebra* A *over a field of characteristic* 0 *with* $\dim A_n < \infty$*. Then the natural map* $\mathrm{U}(P_A) \to A$ *is an isomorphism, where* P_A *is the Lie algebra of primitive elements of* A*. Note this is a graded universal enveloping algebra.*

Theorem 1.5.3 (PBW theorem Theorem C in [4])**.** *The map* Γ *defined as the composition*

$$\mathrm{Sym}(\mathrm{H}(\mathrm{B}\,\mathbb{C}^*))_{vir} \otimes \mathcal{BPS}) \hookrightarrow \mathrm{Free}(\mathrm{H}(\mathrm{B}\,\mathbb{C}^*))_{vir} \otimes \mathcal{BPS}) \hookrightarrow \mathrm{Free}(RA_{W,Q}) \to RA_{W,Q}$$

is an isomorphism. Here the last map is induced by the CoHA multiplication. We obtain in a similar way a map in the absolute case. To summarise we have an isomorphism of algebras and of vector spaces respectively.

$$\mathrm{Sym}(\mathrm{H}(\mathrm{B}\,\mathbb{C}^*))_{vir} \otimes \mathcal{BPS}) \to RA_{Q,W}$$
$$\mathrm{Sym}(\mathrm{H}(\mathrm{B}\,\mathbb{C}^*))_{vir} \otimes \mathrm{BPS}) \to A_{W,Q}.$$

Let us remark that one actually needs to add a sign twist to the CoHA multiplication to make this theorem work.

Proof. We only give the idea here. We can check the isomorphism on fibers $x \in \mathcal{M}_{Q.d}$ for some semisimple rep x. Since $\mathrm{Gr}_{\mathfrak{P}}(A_{W,Q})$ is a Hopf algebra we can consider its Lie algebra of primitives. We can do the same for the fiberwise version, so consider $\mathfrak{g} \subseteq \mathrm{Gr}_{\mathfrak{P}}(A_{W,Q})_x$. Then by the Milnor-Moore theorem we have an isomorphism $\mathrm{U}(g) \cong \mathrm{Gr}_{\mathfrak{P}}(A_{W,Q})_x$. One can then show that we can factor the map Γ_x

$$\begin{array}{ccccc} \mathrm{Sym}(\mathrm{H}(\mathrm{B}\mathbb{C}^*)_{vir} \otimes \mathrm{BPS}_x) & & \xrightarrow{\quad\Gamma_X\quad} & & \\ \big\downarrow & & & & \\ \mathrm{Sym}(\mathfrak{g}) & \xrightarrow[\cong \mathrm{PBW}]{} & \mathrm{U}(\mathfrak{g}) & \xrightarrow[\cong \mathrm{MM}]{} & \mathrm{Gr}_{\mathfrak{P}}(A_{W,Q})_x \end{array}$$

This gives injectivity of Γ_x, now we can use the fibrewise version of the cohomological integrality theorem to deduce that it must be surjective as well. □

Proposition 1.5.4 (Super-commutativity of associated graded)**.** *The associated graded* $\mathrm{Gr}_{\mathfrak{P}}(A_W)$ *to the perverse filtration and the relative CoHA* $RA_{W,Q}$ *are super-commutative algebras.*

Proof. We have an inclusion $\mathfrak{g} = \mathrm{H}(\mathrm{B}\,\mathbb{C}^*) \otimes \mathrm{BPS} \subseteq P$ and an inclusion of algebras $\mathrm{Sym}(\mathfrak{g}) \otimes \mathrm{BPS}) \subseteq \mathrm{Sym}(P) \subseteq \mathrm{U}(P) \cong \mathrm{Gr}_{\mathfrak{P}}(A_{W,Q})$, where P is the space of primitives. Then using cohomological integrality we can prove that the first space has the same graded dimension as the whole associated graded, thus we actually have an equality $\mathfrak{g} = P$. Note that $\mathfrak{g}$ is concentrated in odd perverse degrees, thus when taking the commutator bracket, induced from $\mathrm{Gr}_{\mathfrak{P}}(\mathrm{H}(A_W))$ we get elements in even degree, which forces the bracket to be 0. To sum up we get that $\mathrm{U}(\mathfrak{g})$ is the universal enveloping algebra of an abelian Lie algebra, thus $\mathrm{Gr}_{\mathfrak{P}}(A_{W,Q})$ is super-commutative as well. □

Corollary 1.5.5 (Corollary of PBW and commutativity of associated graded.)**.** $RA_{Q,W}$ *is a super commutative algebra.*

This claim can also be checked on fibres in a similar way and using the previous proposition.

Definition 1.5.6 (BPS Lie algebra)**.** Define

$$\mathfrak{g}_{Q,W} := \mathfrak{P}_1 A_{W,Q} = \mathrm{BPS}[-1].$$

The Lie bracket is well defined because the associated graded $\mathrm{Gr}\,\mathfrak{P}_\bullet A_{W,Q}$ is graded-commutative. This makes the bracket of two elements in $\mathfrak{g}_{Q,W}$ is 0 in the associated graded, so it lives in $\mathfrak{P}_1\,\mathrm{H}(A_W)$.

Note that unlike in the classical case, the CoHA is not the universal enveloping algebra of the BPS Lie algebra. However, it turns out we always have an inclusion.

Proposition 1.5.7 (Proposition 2.6 in [1])**.** *There is an inclusion of algebras*

$$\mathrm{U}(\mathfrak{g}_Q) \to A_{W,Q}.$$

Example 1.5.8 (BPS Lie algebra for symmetric quiver without potential)**.** By Efimovs theorem [5], the CoHA of a symmetric quiver with potential is supercommutative. We can interpret this as saying that the PBW isomorphism is an isomorphism of algebras in this case. In particular, this also means that the BPS Lie algebra is abelian.

Example 1.5.9 (BPS Lie algebra for tripled Jordan quiver)**.** We want to return to example 1.4.4 and compute the BPS Lie algebra. We already know that $\mathcal{BPS} = \bigoplus_d \Delta^d_* \mathbb{Q}_{\mathbb{A}^3}[3]$. Now we want to pushforward to the point to get $\mathrm{BPS} = \bigoplus_d \mathrm{H}(\mathbb{A}^3, \mathbb{Q})[3] = \bigoplus \mathbb{Q}[3]$. Hence we get $\mathfrak{g}_Q = \bigoplus_d \mathbb{Q}[2]$. So the entire BPS Lie algebra is concentrated in cohomological degree 2. Now the commutator bracket is compatible with cohomological degree so the bracket of any two elements is in cohomological degree 4, hence the bracket must be 0. We see completely formally then, that the BPS Lie algebra must be abelian. With more work, one can show that actually the CoHA is not supercommutative, which is different from the analogy to Lie algebras. If a Lie algebra is abelian then the universal enveloping algebra is a polynomial algebra.

1.6 Relation to 2d CoHA and non-abelian examples

So far all our examples have been abelian. In this section we explore a relation to the preprojective CoHA and relate the BPS Lie algebra to Lie algebras well known in representation theory. Let us start by briefly giving definitions of the 2d story. We will only give the main theorems without going into full detail. We follow [1]. Let us recall some of the set up. Here we denote by Π_Q the preprojective algebra associated to a quiver Q and $\mathfrak{M}_{\Pi_Q}$ its stack of representations and $\mathcal{M}_{\Pi_Q}$ the corresponding coarse moduli space.

Definition 1.6.1 (2d CoHA)**.** Define the relative $2d$ CoHA $RA_{\Pi_Q} = \mathrm{JH}_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\Pi_Q}}$. Here JH is the semisimplification morphism $\mathfrak{M}_{\Pi_Q} \to \mathcal{M}_{\Pi_Q}$. Define the absolute $2d$ CoHA $A_{\Pi_Q} = \mathrm{H}(\mathfrak{M}_{\Pi_Q}, \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\Pi_Q}})$

Theorem 1.6.2. *The relative* $2d$ *CoHA splits*

$$RA_{\Pi_Q} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} {}^p\mathcal{H}^n(RA_{\Pi_Q})$$

This induces a less perverse filtration starting in degree 0 *on the absolute* $2d$ *CoHA* $\mathfrak{L}_k = \mathrm{H}(\mathcal{M}_{\Pi_Q}, \tau^{\leq i} RA_{\Pi_Q})$ *The multiplication of the preprojective CoHA preserves the less perverse filtration. Furthermore, we have an isomorphism of algebras*

$$\mathfrak{L}_0\,\mathrm{H}(A_{\Pi_Q}) \cong \mathrm{U}(\mathfrak{g}_{\Pi_Q})$$

where $\mathfrak{g}_{\Pi_Q} \cong \mathfrak{g}_{\widetilde{Q}}$

We can relate the critical CoHA to the preprojective CoHA by using the tripling construction for a quiver and a dimensional reduction theorem. If Q is a quiver then the tripled quiver is given by taking the double $\bar{Q}$ and then adding a loop at each vertex. We then consider the potential $\widetilde{W} = \sum_{a \in Q_1} [a, a^*] \sum_{i \in Q_0} \omega_i$ where ω_i are the added loops. We have seen a basic example of this construction for the Jordan quiver in 1.4.4.

Theorem 1.6.3 (Dimensional reduction)**.** *Let* $\pi : \mathfrak{M}_{\widetilde{Q}} \to \mathfrak{M}_{\bar{Q}}$ *be the forgetful morphism. By the dimensional reduction theorem [2] we have an isomorphism* $\pi_* \phi_{Tr(\widetilde{W})} \mathbb{Q}_{\mathfrak{M}_{\widetilde{Q}},vir} \cong \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\Pi_Q},vir}$*. Therefore we get an isomorphism* $A_{\widetilde{W},\widetilde{Q}} \cong A_{\Pi_Q}$*.*

Using this isomorphism, the preprojective CoHA also inherits the perverse filtration discussed previously.

Theorem 1.6.4 (Kac-Moody and BPS Lie algebras)**.** *Let Q be a quiver. We have an isomorphism of algebras*

$$\mathrm{U}(\mathfrak{n}^-_{Q'}) \cong \mathfrak{L}_0 \, \mathrm{H}^0(A_{\Pi_Q})$$

here Q' is the real subquiver of Q and $\mathfrak{n}^-_{Q'}$ is the negative part of the Kac-Moody Lie algebra for Q'. This also restricts to an isomorphism

$$\mathfrak{n}^-_{Q'} \cong \mathrm{H}^0(\mathfrak{g}_{\Pi_Q})$$

Here the real subquiver Q' is the quiver Q with the vertices that have loops removed.

Theorem 1.6.5 (BPS Lie algebras for affine ADE quivers)**.** *Let Q be an affine ADE quiver. We have an isomorphism of Lie algebras.*

$$\mathfrak{g}_{\pi_Q} \cong \mathfrak{n}^-_{Q'} \oplus s\mathbb{Q}[s]$$

s^n lives in dimension degree $n\delta$ and cohomological degree -2, where δ is the unique primitive imaginary root of Q.

References

[1] Ben Davison. "BPS Lie algebras and the less perverse filtration on the preprojective CoHA". In: *arXiv preprint arXiv:2007.03289* (2020).

[2] Ben Davison. "The critical CoHA of a quiver with potential". In: *Quarterly Journal of Mathematics* 68.2 (2017), pp. 635–703.

[3] Ben Davison. "The integrality conjecture and the cohomology of preprojective stacks". In: *arXiv preprint arXiv:1602.02110* (2016).

[4] Ben Davison and Sven Meinhardt. "Cohomological Donaldson–Thomas theory of a quiver with potential and quantum enveloping algebras". In: *Inventiones mathematicae* 221 (2020), pp. 777–871.

[5] Alexander I Efimov. "Cohomological Hall algebra of a symmetric quiver". In: *Compositio Mathematica* 148.4 (2012), pp. 1133–1146.

[6] Sven Meinhardt and Markus Reineke. "Donaldson–Thomas invariants versus intersection cohomology of quiver moduli". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2019.754 (2019), pp. 143–178.