The critical cohomological Hall algebra of a quiver with potential

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The goal of this talk is to build a Hall algebra structure on the critical, equivariant cohomology of moduli of quiver representations. Critical cohomology takes as input a regular function on quiver moduli, which is derived from a potential W.

We will build a Hall algebra product and a coproduct on this critical cohomology. The algebra structure generalizes the Hall algebras associated to a symmetric quiver or a preprojective algebra, which were introduced in previous talks.

1 Quivers with potential and their moduli

1.1 Basic notions

Let $Q = (Q_0, Q_1, s, t)$ be a quiver.

Definition 1.1. A potential of Q is a formal $\mathbb{C}$-linear combination W of oriented cycles of Q, presented up to cyclic permutation.

Example 1.2. Let Q be the quiver with one vertex and three loops x, y, z. Then $W = [xyz] - [xzy] = [yzx] - [yxz]$ is a potential.

We will build a CoHA from the cohomology of the following quiver moduli. Let us consider the representation space

$$X_{\mathbf{d}} := \prod_{\substack{a \in Q_1 \\ a: i \to j}} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\oplus \mathbf{d}_i}, \mathbb{C}^{\oplus \mathbf{d}_j})$$

endowed with the natural action of

$$\mathrm{GL}_{\mathbf{d}} := \prod_{i \in Q_0} \mathrm{GL}(\mathbf{d}_i, \mathbb{C}),$$

where $\mathbf{d} \in \mathbb{Z}_{\geqslant 0}^{Q_0}$ is a dimension vector.

Definition 1.3. The moduli stack of representations of Q is the following disjoint union of quotient stacks:

$$\mathfrak{M} = \bigsqcup_{\mathbf{d} \in \mathbb{Z}_{\geqslant 0}^{Q_0}} \mathfrak{M}_{\mathbf{d}} := \bigsqcup_{\mathbf{d} \in \mathbb{Z}_{\geqslant 0}^{Q_0}} X_{\mathbf{d}} / \mathrm{GL}_{\mathbf{d}}\,.$$

The good moduli space of representations of Q is the following disjoint union of schemes:

$$\mathcal{M} = \bigsqcup_{\mathbf{d}\in\mathbb{Z}_{\geqslant 0}^{Q_0}} \mathcal{M}_{\mathbf{d}} := \bigsqcup_{\mathbf{d}\in\mathbb{Z}_{\geqslant 0}^{Q_0}} \operatorname{Spec}(\mathbb{C}[X_{\mathbf{d}}]^{\mathrm{GL}_{\mathbf{d}}}).$$

.

$\mathbb{C}$-points of $\mathfrak{M}$ (resp. $\mathcal{M}$) parametrize finite-dimensional representations of Q (resp. semisimple representations of Q). There is a canonical morphism $p : \mathfrak{M} \to \mathcal{M}$ mapping a representation to its semisimplification.

Remark. There are variants of $\mathfrak{M}$ (resp. $\mathcal{M}$) parametrizing semistable (resp. polystable) representations of Q for a given stability condition (see [7]). One can also work with the substack of representations satisfying a property S (S should be stable under extensions, in order to build a Hall algebra product). We leave these variants out to simplify the discussion.

In order to define critical cohomology (see section 2), we will use a regular function $\operatorname{Tr}(W) : \mathcal{M} \to \mathbb{A}^1_{\mathbb{C}}$ attached to a potential:

Definition 1.4. Let W be a potential of Q, presented as follows:

$$W = \sum_c \alpha_c \cdot [a_{c,1} \dots a_{c,l_c}],$$

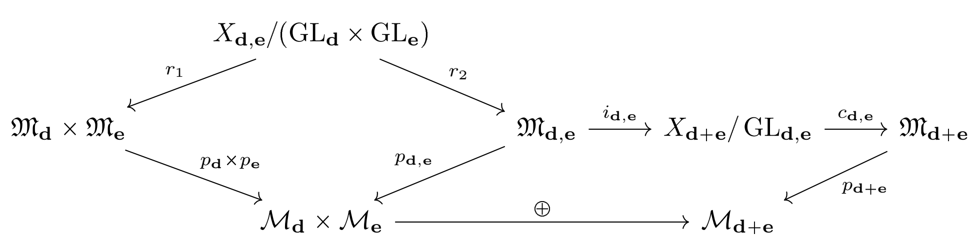
where $\alpha_c \in \mathbb{C}$ and $a_{c,i} \in Q_1$ for $1 \leqslant i \leqslant l_c$. Then the following expression yields a well-defined, $\mathrm{GL}_{\mathbf{d}}$-invariant function $\operatorname{Tr}(W)$ on $X_{\mathbf{d}}$:

$$\operatorname{Tr}(W)(x) = \sum_c \alpha_c \cdot \operatorname{Tr}(x_{a_{c,1}} \dots x_{a_{c,l_c}}),\ x \in X_{\mathbf{d}}.$$

Note that $\operatorname{Tr}(W)$ descends to a regular function on $\mathcal{M}$, as it is $\mathrm{GL}_{\mathbf{d}}$-invariant.

1.2 Correspondences for the Hall product

We now turn to the correspondence diagram which we will use to build the product and coproduct:



Here, $\mathfrak{M}_{\mathbf{d},\mathbf{e}}$ is the stack parametrizing short exact sequences $0 \to M' \to M \to M'' \to 0$ of representations of Q where $\dim(M') = \mathbf{d}$ and $\dim(M'') = \mathbf{e}$. $\mathfrak{M}_{\mathbf{d},\mathbf{e}}$ can be presented as a quotient stack $\mathfrak{M}_{\mathbf{d},\mathbf{e}} = X_{\mathbf{d},\mathbf{e}}/\mathrm{GL}_{\mathbf{d},\mathbf{e}}$, where $X_{\mathbf{d},\mathbf{e}} \subseteq X_{\mathbf{d}+\mathbf{e}}$ (resp. $\mathrm{GL}_{\mathbf{d},\mathbf{e}} \subseteq \mathrm{GL}_{\mathbf{d}+\mathbf{e}}$) is the subspace (resp. subgroup) of elements stabilizing the graded vector space $\mathbb{C}^{\oplus\mathbf{e}} \subseteq \mathbb{C}^{\oplus\mathbf{d}}$.

Let $\chi(\mathbf{d},\mathbf{e}) = \mathbf{d}\cdot\mathbf{e} - \sum_{a\in Q_1} d_{s(a)}e_{t(a)} = \sum_{i\in Q_0} d_ie_i - \sum_{a\in Q_1} d_{s(a)}e_{t(a)}$. Then:

- r_1 is induced by the forgetful morphism $X_{\mathbf{d},\mathbf{e}} \to X_{\mathbf{d}} \times X_{\mathbf{e}}$. It is an affine fibration of relative dimension $\mathbf{d} \cdot \mathbf{e} - \chi(\mathbf{e}, \mathbf{d})$;

- r_2 is induced by the morphism $\mathrm{GL}_{\mathbf{d}} \times \mathrm{GL}_{\mathbf{e}} \to \mathrm{GL}_{\mathbf{d},\mathbf{e}}$. It is an affine fibration of relative dimension $\mathbf{d} \cdot \mathbf{e}$;

- $i_{\mathbf{d},\mathbf{e}}$ is induced by $X_{\mathbf{d},\mathbf{e}} \subseteq X_{\mathbf{d}+\mathbf{e}}$. It is a closed immersion and $\dim X_{\mathbf{d}+\mathbf{e}} - \dim X_{\mathbf{d},\mathbf{e}} = \mathbf{d} \cdot \mathbf{e} - \chi(\mathbf{d}, \mathbf{e})$;

- $c_{\mathbf{d},\mathbf{e}}$ is induced by $\mathrm{GL}_{\mathbf{d},\mathbf{e}} \subseteq \mathrm{GL}_{\mathbf{d}+\mathbf{e}}$. It is a proper map and $\dim \mathrm{GL}_{\mathbf{d}+\mathbf{e}} - \dim \mathrm{GL}_{\mathbf{d},\mathbf{e}} = \mathbf{d} \cdot \mathbf{e}$.

2 Critical cohomology, vanishing cycles and Thom-Sebastiani isomorphism

The underlying ($\mathbb{Z} \times \mathbb{Z}_{\geqslant 0}^{Q_0}$-graded) vector space of the critical CoHA is:

$$\bigoplus_{\mathbf{d} \in \mathbb{Z}_{\geqslant 0}^{Q_0}} \mathrm{H}_c^{\bullet}(\mathfrak{M}_{\mathbf{d}}, \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{\mathfrak{M}_{\mathbf{d}}}[-1])^{\vee} [\chi(\mathbf{d}, \mathbf{d})].$$

Contrary to the CoHAs introduced in the previous talks, cohomology of quiver moduli is taken with respect to a constructible complex of sheaves which we now introduce.

2.1 Constructible complexes and (hyper)cohomology

Given a complex variety X, we work within the category of (bounded below) constructible complexes $D_c^+(X)$ (see [6] for an introduction). Objects in this category are complexes $\mathcal{F}^{\bullet}$ of sheaves of $\mathbb{Q}$-vector spaces on X. We will often consider the total cohomology of $\mathcal{F}^{\bullet}$ as a complex of sheaves:

$$\mathcal{H}(\mathcal{F}^{\bullet}) = \bigoplus_i \mathcal{H}^i(\mathcal{F}^{\bullet})[-i].$$

Given a morphism $p : X \to Y$ of complex varieties, we will use the derived pushforward functors $p_* = \mathrm{R}p_* : D_c^+(X) \to D_c^+(Y)$ and $p_! = \mathrm{R}p_!$ and the (exact) pullback functor $p^{-1} : D_c^+(Y) \to D_c^+(X)$. We will also make use of the derived tensor product $\bullet \otimes \bullet = \bullet \overset{\mathrm{L}}{\otimes} \bullet : D_c^+(X) \times D_c^+(X) \to D_c^+(X)$ and the external tensor product $\bullet \boxtimes \bullet = (pr_X^{-1} \bullet) \otimes (pr_Y^{-1} \bullet) : D_c^+(X) \times D_c^+(Y) \to D_c^+(X \times Y)$.

Definition 2.1. Let $\mathcal{F}^{\bullet} \in D_c^+(X)$ and $p : X \to \mathrm{pt}$. The hypercohomology of X (with compact support) with respect to $\mathcal{F}^{\bullet}$ is:

$$\mathrm{H}^{\bullet}(X, \mathcal{F}^{\bullet}) := \mathcal{H}(p_* \mathcal{F}^{\bullet}),$$

$$\mathrm{H}_c^{\bullet}(X, \mathcal{F}^{\bullet}) := \mathcal{H}(p_! \mathcal{F}^{\bullet}).$$

2.2 Vanishing cycles and Thom-Sebastiani isomorphism

Let X be a smooth complex variety and $f : X \to \mathbb{C}$ be a regular function on X. A famous result of Milnor and Lê asserts that, for $x \in f^{-1}(0)$ and $0 < \delta << \epsilon << 1$, $f : B_{\epsilon}(x) \cap f^{-1}(D_{\delta}^*) \to D_{\delta}^*$ is a smooth, locally trivial fibration, where $D_\delta^* = \{z \in \mathbb{C} \mid 0 < |z| < \delta\}$ and $B_\epsilon(x)$ is an open ball of radius ϵ, centered at x. The fiber $F_{f,x}$ is called the Milnor fiber of f at x (see for instance [5]).

Attached to f are two functors $\psi_f, \phi_f : D_c^+(X) \to D_c^+(X)$ called functors of nearby cycles and vanishing cycles, which are formally defined as follows:

Definition 2.2. Let $j : f^{-1}(\{\mathrm{Re}(z) < 0\}) \hookrightarrow X$ and $i : f^{-1}(0) \hookrightarrow X$ and $\mathcal{F}^\bullet \in D_c^+(X)$. Then:

$$\psi_f(\mathcal{F}^\bullet) := i_* i^{-1} j_* j^{-1} \mathcal{F}^\bullet,$$

$$\phi_f(\mathcal{F}^\bullet) := \mathrm{cone}(i_* i^{-1} \mathcal{F}^\bullet \to \psi_f(\mathcal{F}^\bullet)).$$

The assignment $\mathcal{F}^\bullet \mapsto \phi_f(\mathcal{F}^\bullet)$ can be made into a functor, despite being defined as a cone. We call $\mathrm{H}^\bullet(X, \phi_f \underline{\mathbb{Q}}_X)$ the critical cohomology of (X, f).

The stalks of these complexes encode local information on f, as they compute the cohomology of Milnor fibers (see [5, Ch. 4.2.]). For $x \in f^{-1}(0)$:

$$\mathcal{H}(\psi_f(\mathcal{F}^\bullet))_x \simeq \mathrm{H}^\bullet(F_{f,x}, \mathcal{F}^\bullet),$$

$$\mathcal{H}(\phi_f(\mathcal{F}^\bullet))_x \simeq \mathrm{H}^\bullet(B_\epsilon(x) \backslash F_{f,x}, F_{f,x}; \mathcal{F}^\bullet)[1].$$

Remark. The motivation for considering (compactly supported) critical cohomology $\mathrm{H}_c^\bullet(\mathfrak{M}_{\mathbf{d}}, \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}[-1])$ comes from the following fact: when a moduli space is (locally) defined as the critical locus of a regular function f in a smooth ambient space X, the Euler characteristic of critical cohomology computes the degree of its virtual fundamental class (assuming it exists, see [1]). The critical locus of $\mathrm{Tr}(W)$ in $\mathfrak{M}_{\mathbf{d}}$ can be seen a local model for moduli spaces of objects in a 3-Calabi-Yau category (see [8]).

In order to construct the Hall algebra product, we will need to combine critical cohomology of various components of $\mathfrak{M}$. This is done using the Thom-Sebastiani isomorphism.

Theorem 2.3 (Thom-Sebastiani isomorphism)**.** *Consider X and Y two smooth complex varieties and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$ two regular functions. Let $f \boxplus g : X \times Y \to \mathbb{C}$ be defined by $(f \boxplus g)(x, y) = f(x) + g(y)$ and $k : f^{-1}(0) \times g^{-1}(0) \hookrightarrow (f \boxplus g)^{-1}(0)$. Let also $\mathcal{F}^\bullet \in D_c^+(X)$ and $\mathcal{G}^\bullet \in D_c^+(Y)$. Then:*

$$k^{-1} \phi_{f \boxplus g}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)[-1] \simeq \phi_f \mathcal{F}^\bullet[-1] \boxtimes \phi_g \mathcal{G}^\bullet[-1].$$

Remark. As explained in [4, Prop. 2.13.], the Thom-Sebastiani isomorphism can be upgraded to *monodromic* mixed Hodge modules. The CoHA can be built out of graded monodromic mixed Hodge structures instead of graded vector spaces. We leave these details out to avoid technicalities related to monodromic mixed Hodge modules.

2.3 Equivariant critical cohomology

We introduced above critical cohomology for schemes. We will need critical cohomology for quotient stacks $X_{\mathbf{d}} / \mathrm{GL}_{\mathbf{d}}$ in order to build the critical CoHA. In [4, §2.2.] and [3, §2.1.3.], a recipe is developed to build the complex $p_* \phi_f \underline{\mathbb{Q}}_{X/G}$ - or its total *perverse* cohomology for any morphism $p : X/G \to Y$, where X is a smooth complex G-variety, f

is a G-invariant function and Y is a complex variety. Here, we will only give its relevant functorial properties for building the CoHA product and coproduct.

We first give general results on $p_*\phi_f\underline{\mathbb{Q}}_{X/G}$:

Proposition 2.4. *Let X, X' be smooth complex G-varieties, $h : X' \to X$ be a G-equivariant morphism and $f : X \to \mathbb{C}$ a regular function. We also call $h : X'/G \to X/G$ the induced morphism of stacks. Then:*

(i) If h is an affine fibration, then there is a natural isomorphism

$$p_*\phi_f\underline{\mathbb{Q}}_{X/G} \to (p \circ h)_*\phi_{f\circ h}\underline{\mathbb{Q}}_{X'/G}.$$

(ii) If h is proper, then there are natural morphisms

$$p_*\phi_f\underline{\mathbb{Q}}_{X/G} \to (p \circ h)_*\phi_{f\circ h}\underline{\mathbb{Q}}_{X'/G},$$

$$(p \circ h)_*\phi_{f\circ h}\underline{\mathbb{Q}}_{X'/G} \to p_*\phi_f\underline{\mathbb{Q}}_{X/G}[2(\dim X - \dim X')].$$

Let H be an algebraic subgroup of G and consider $h : X/H \to X/G$. Then:

(iii) If G/H is an affine space, then there is a natural isomorphism

$$p_*\phi_f\underline{\mathbb{Q}}_{X/G} \to (p \circ h)_*\phi_{f\circ h}\underline{\mathbb{Q}}_{X/H}.$$

(iv) If G/H is proper, then there are natural morphisms

$$p_*\phi_f\underline{\mathbb{Q}}_{X/G} \to (p \circ h)_*\phi_{f\circ h}\underline{\mathbb{Q}}_{X/H},$$

$$(p \circ h)_*\phi_{f\circ h}\underline{\mathbb{Q}}_{X/H} \to p_*\phi_f\underline{\mathbb{Q}}_{X/G}[2(\dim H - \dim G)].$$

We further give analogous results for equivariant compactly supported critical cohomology:

Proposition 2.5. *Let X, X' be smooth complex G-varieties, $h : X' \to X$ be a G-equivariant morphism and $f : X \to \mathbb{C}$ a regular function. Then:*

(i) If h is an affine fibration, then there is a natural isomorphism

$$\mathrm{H}_c^\bullet(X'/G, \phi_{f\circ h}\underline{\mathbb{Q}}_{X'/G}) \to \mathrm{H}_c^\bullet(X/G, \phi_f\underline{\mathbb{Q}}_{X/G})[2(\dim X - \dim X')].$$

(ii) If h is proper, then there are natural morphisms

$$\mathrm{H}_c^\bullet(X/G, \phi_f\underline{\mathbb{Q}}_{X/G}) \to \mathrm{H}_c^\bullet(X'/G, \phi_{f\circ h}\underline{\mathbb{Q}}_{X'/G}),$$

$$\mathrm{H}_c^\bullet(X'/G, \phi_{f\circ h}\underline{\mathbb{Q}}_{X'/G}) \to \mathrm{H}_c^\bullet(X/G, \phi_f\underline{\mathbb{Q}}_{X/G})[2(\dim X - \dim X')].$$

Let H be an algebraic subgroup of G and consider $h : X/H \to X/G$. Then:

(iii) If G/H is an affine space, then there is a natural isomorphism

$$\mathrm{H}^\bullet_c(X/H, \phi_{f\circ h}\underline{\mathbb{Q}}_{X/H}) \to \mathrm{H}^\bullet_c(X/G, \phi_f\underline{\mathbb{Q}}_{X/G})[2(\dim H - \dim G)].$$

(iv) If G/H is proper, then there is a natural morphism

$$\mathrm{H}^\bullet_c(X/G, \phi_f\underline{\mathbb{Q}}_{X/G}) \to \mathrm{H}^\bullet_c(X/H, \phi_{f\circ h}\underline{\mathbb{Q}}_{X/H}),$$

$$\mathrm{H}^\bullet_c(X/H, \phi_{f\circ h}\underline{\mathbb{Q}}_{X/H}) \to \mathrm{H}^\bullet_c(X/G, \phi_f\underline{\mathbb{Q}}_{X/G})[2(\dim H - \dim G)].$$

We finally give equivariant versions of the Thom-Sebastiani isomorphism:

Proposition 2.6. *Suppose X_i is a smooth complex G_i-variety and f_i is a G_1-invariant regular function on X_i, for $i = 1, 2$. Consider as before a morphism $p_i : X_i/G_i \to Y_i$. Suppose that p_i is proper (or approximated by proper morphisms as in [4, §4.1.]). Then:*

$$(p_1)_*\phi_{f_1}\underline{\mathbb{Q}}_{X_1/G_1}[-1] \boxtimes (p_2)_*\phi_{f_2}\underline{\mathbb{Q}}_{X_2/G_2}[-1] \simeq (p_1 \times p_2)_*\phi_{f_1 \boxplus f_2}\underline{\mathbb{Q}}_{(X_1\times X_2)/(G_1\times G_2)}[-1].$$

Moreover:

$$\mathrm{H}^\bullet_c(X_1/G_1, \phi_{f_1}\underline{\mathbb{Q}}_{X_1/G_1}[-1]) \otimes \mathrm{H}^\bullet_c(X_2/G_2, \phi_{f_2}\underline{\mathbb{Q}}_{X_2/G_2}[-1]) \simeq \mathrm{H}^\bullet_c((X_1\times X_2)/(G_1\times G_2), \phi_{f_1\boxplus f_2}\underline{\mathbb{Q}}_{(X_1\times X_2)/(G_1\times G_2)}[-1]).$$

3 Relative and absolute CoHA products

We now define the CoHA product on $\mathrm{H}_{Q,W} := \bigoplus_{\mathbf{d}\in\mathbb{Z}^{Q_0}_{\geqslant 0}} \mathrm{H}^\bullet_c(\mathfrak{M}_{\mathbf{d}}, \phi_{\mathrm{Tr}(W)}\underline{\mathbb{Q}}_{\mathfrak{M}_{\mathbf{d}}}[-1])^\vee[\chi(\mathbf{d},\mathbf{d})]$. We also upgrade this product to an algebra object in the category of constructible complexes of sheaves on $\mathcal{M}$ (the relative CoHA), as this is a key ingredient for proving structure results on the usual (absolute) CoHA (see following talk).

3.1 Monoidal structures and relative CoHA

From now on, we assume that Q is symmetric.

We first shift perspectives on the absolute CoHA product, in order to introduce the relative CoHA product. As we will see below, $\mathrm{H}_{Q,W}$ is a $\mathbb{Z}\times\mathbb{Z}^{Q_0}_{\geqslant 0}$-graded algebra i.e. for $\mathbf{d}, \mathbf{e} \in \mathbb{Z}^{Q_0}_{\geqslant 0}$, we have a degree-preserving map of $\mathbb{Z}$-graded vector spaces:

$$\mathrm{H}_{Q,W,\mathbf{d}} \otimes \mathrm{H}_{Q,W,\mathbf{e}} \to \mathrm{H}_{Q,W,\mathbf{d}+\mathbf{e}}.$$

Seeing $\mathrm{H}_{Q,W,\mathbf{d}}$ (resp. $\mathrm{H}_{Q,W,\mathbf{e}}$) as a constructible complex on $\mathrm{pt}_{\mathbf{d}}$ (resp. $\mathrm{pt}_{\mathbf{e}}$), we can reinterpret this morphism as a morphism of complexes on $\mathrm{pt}_{\mathbf{d}+\mathbf{e}}$:

$$+_*\,(\mathrm{H}_{Q,W,\mathbf{d}} \boxtimes \mathrm{H}_{Q,W,\mathbf{e}}) \to \mathrm{H}_{Q,W,\mathbf{d}+\mathbf{e}},$$

where $+ : \mathrm{pt}_{\mathbf{d}} \times \mathrm{pt}_{\mathbf{e}} \to \mathrm{pt}_{\mathbf{d}+\mathbf{e}}$ is the natural isomorphism.

This leads us to consider $\mathrm{H}_{Q,W}$ as an object in $D^+_c(\mathbb{Z}^{Q_0}_{\geqslant 0})$ and the CoHA product as a morphism:

$$\mathrm{m} : +_* (\mathrm{H}_{Q,W} \boxtimes \mathrm{H}_{Q,W}) \to \mathrm{H}_{Q,W},$$

where $+ : \mathbb{Z}^{Q_0}_{\geqslant 0} \times \mathbb{Z}^{Q_0}_{\geqslant 0} \to \mathbb{Z}^{Q_0}_{\geqslant 0}$ is the addition morphism.

Upgrading this to $D^+_c(\mathcal{M})$, we define:

$$\mathcal{H}_{Q,W} := \bigoplus_{\mathbf{d} \in \mathbb{Z}^{Q_0}_{\geqslant 0}} (p_{\mathbf{d}})_* \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{\mathfrak{M}_{\mathbf{d}}}[-1][-\chi(\mathbf{d},\mathbf{d})]$$

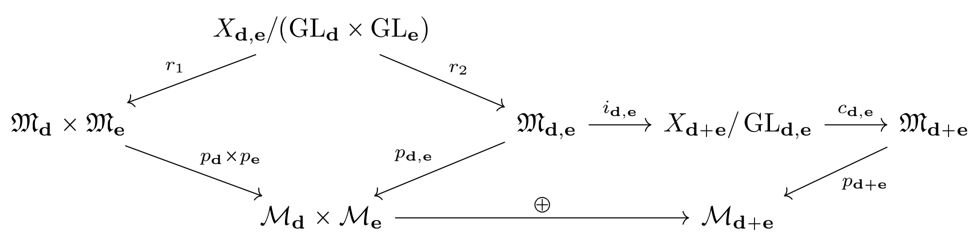
and we define in the following section a *relative* CoHA product:

$$m : \oplus_*(\mathcal{H}_{Q,W} \boxtimes \mathcal{H}_{Q,W}) \to \mathcal{H}_{Q,W}.$$

Remark. As discussed in [4, Rem. 3.3.], the assumption that Q is symmetric may be weakened to some extent when working with stability conditions. We leave out this discussion to simplify computations.

3.2 Construction of products

Recall the diagram of correspondences:



We are now ready to define the relative and absolute CoHA products. We define the restriction of m to summands $\oplus_*(\mathcal{H}_{Q,W,\mathbf{d}} \boxtimes \mathcal{H}_{Q,W,\mathbf{e}})$ as the composition of the following morphisms, shifted by $-\chi(\mathbf{d},\mathbf{d}) - \chi(\mathbf{e},\mathbf{e})$:

$$\begin{aligned}
\oplus_*(\mathcal{H}_{Q,W,\mathbf{d}}[\chi(\mathbf{d},\mathbf{d})] \boxtimes \mathcal{H}_{Q,W,\mathbf{e}}[\chi(\mathbf{e},\mathbf{e})]) &\simeq \oplus_*(p_{\mathbf{d}} \times p_{\mathbf{e}})_* \phi_{\mathrm{Tr}(W)_{\mathbf{d}} \boxplus \mathrm{Tr}(W)_{\mathbf{e}}} \underline{\mathbb{Q}}_{\mathfrak{M}_{\mathbf{d}} \times \mathfrak{M}_{\mathbf{e}}}[-1] && \text{(Thom-Sebastiani)} \\
&\simeq \oplus_*((p_{\mathbf{d}} \times p_{\mathbf{e}}) \circ r_1)_* \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{X_{\mathbf{d},\mathbf{e}}/(\mathrm{GL}_{\mathbf{d}} \times \mathrm{GL}_{\mathbf{e}})}[-1] && \text{(2.4.i.)} \\
&\simeq \oplus_*(p_{\mathbf{d},\mathbf{e}} \circ r_2)_* \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{X_{\mathbf{d},\mathbf{e}}/(\mathrm{GL}_{\mathbf{d}} \times \mathrm{GL}_{\mathbf{e}})}[-1] \\
&\simeq \oplus_*(p_{\mathbf{d},\mathbf{e}})_* \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{\mathfrak{M}_{\mathbf{d},\mathbf{e}}}[-1] && \text{(2.4.iii.)} \\
&\simeq (p_{\mathbf{d}} \circ c_{\mathbf{d},\mathbf{e}} \circ i_{\mathbf{d},\mathbf{e}})_* \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{\mathfrak{M}_{\mathbf{d},\mathbf{e}}}[-1] \\
&\to (p_{\mathbf{d}} \circ c_{\mathbf{d},\mathbf{e}})_* \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{X_{\mathbf{d}+\mathbf{e}}/\mathrm{GL}_{\mathbf{d},\mathbf{e}}}[-1][2(\mathbf{d} \cdot \mathbf{e} - \chi(\mathbf{d},\mathbf{e}))] && \text{(2.4.ii.)} \\
&\to (p_{\mathbf{d}})_* \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{\mathfrak{M}_{\mathbf{d}}}[-1][-2\chi(\mathbf{d},\mathbf{e})]. && \text{(2.4.iv.)}
\end{aligned}$$

Note that the overall shift of the right-hand side is $-\chi(\mathbf{d},\mathbf{d}) - \chi(\mathbf{e},\mathbf{e}) - 2\chi(\mathbf{d},\mathbf{e}) = -\chi(\mathbf{d}+\mathbf{e},\mathbf{d}+\mathbf{e})$, as Q is symmetric. Likewise we define the restriction of m to the summand $\mathrm{H}_{Q,W,\mathbf{d}} \otimes \mathrm{H}_{Q,W,\mathbf{e}}$ as the following morphism, shifted by $\chi(\mathbf{d},\mathbf{d}) + \chi(\mathbf{e},\mathbf{e})$ (mind dualizations):

$$
\begin{aligned}
\mathrm{H}_{Q,W,\mathbf{d}}[-\chi(\mathbf{d},\mathbf{d})] \otimes \mathrm{H}_{Q,W,\mathbf{e}}[-\chi(\mathbf{e},\mathbf{e})] &\simeq \mathrm{H}_c^\bullet(\mathfrak{M}_\mathbf{d} \times \mathfrak{M}_\mathbf{e}, \phi_{\mathrm{Tr}(W)_\mathbf{d} \boxplus \mathrm{Tr}(W)_\mathbf{e}} \underline{\mathbb{Q}}_{\mathfrak{M}_\mathbf{d} \times \mathfrak{M}_\mathbf{e}}[-1])^\vee && \text{(Thom-Sebastiani)} \\
&\simeq \mathrm{H}_c^\bullet(X_{\mathbf{d},\mathbf{e}}/(\mathrm{GL}_\mathbf{d} \times \mathrm{GL}_\mathbf{e}), \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{X_{\mathbf{d},\mathbf{e}}/(\mathrm{GL}_\mathbf{d} \times \mathrm{GL}_\mathbf{e})}[-1])^\vee [2(\chi(\mathbf{e},\mathbf{d}) - \mathbf{d}\cdot\mathbf{e})] && \text{(2.5.i.)} \\
&\simeq \mathrm{H}_c^\bullet(\mathfrak{M}_{\mathbf{d},\mathbf{e}}, \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{\mathfrak{M}_{\mathbf{d},\mathbf{e}}}[-1])^\vee [2\chi(\mathbf{e},\mathbf{d})] && \text{(2.5.iii.)} \\
&\to \mathrm{H}_c^\bullet(X_{\mathbf{d}+\mathbf{e}}/\mathrm{GL}_{\mathbf{d},\mathbf{e}}, \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{X_{\mathbf{d}+\mathbf{e}}/\mathrm{GL}_{\mathbf{d},\mathbf{e}}}[-1])^\vee [2\chi(\mathbf{e},\mathbf{d})] && \text{(2.5.ii.)} \\
&\to \mathrm{H}_c^\bullet(\mathfrak{M}_{\mathbf{d}+\mathbf{e}}, \phi_{\mathrm{Tr}(W)} \underline{\mathbb{Q}}_{\mathfrak{M}_{\mathbf{d}+\mathbf{e}}}[-1])^\vee [2\chi(\mathbf{e},\mathbf{d})] && \text{(2.5.iv.)}
\end{aligned}
$$

We have the following results:

Theorem 3.1. *([8, §7.6.], [4, Prop. 5.1.]) The products* m *and* m *are associative.*

4 Construction of a coproduct

In this final part, we define a *localised* coproduct on the absolute CoHA. In order to build a comultiplication which is compatible with the product above, one first needs to localize the Hall algebra and define some signs conventions.

4.1 Localised bialgebras

The natural target of a coproduct on $\mathrm{H}_{Q,W}$ is:

$$\mathrm{H}_{Q,W} \boxtimes_+ \mathrm{H}_{Q,W} := +_*(\mathrm{H}_{Q,W} \boxtimes \mathrm{H}_{Q,W}) = \bigoplus_{\mathbf{d} \in \mathbb{Z}_{\geqslant 0}^{Q_0}} \bigoplus_{\mathbf{d}=\mathbf{e}+\mathbf{f}} \mathrm{H}_{Q,W,\mathbf{e}} \otimes \mathrm{H}_{Q,W,\mathbf{f}}.$$

However, in order to obtain a well-behaved comultiplication, one needs to localize this target with respect to the actions of some symmetric algebras. Indeed, $\mathrm{H}_{Q,W,\mathbf{d}}$ is naturally endowed with an action of:

$$\mathrm{H}_{\mathrm{GL}_\mathbf{d}}(\mathrm{pt}) = \mathbb{Q}\left[x_{i,k},\ {}^{i \in Q_0}_{1 \leqslant k \leqslant \mathbf{d}_i}\right]^{\mathcal{S}_\mathbf{d}},$$

where $\mathcal{S}_\mathbf{d} := \prod_{i \in Q_0} \mathcal{S}_{\mathbf{d}_i}$. The target of the comultiplication is then:

$$\mathrm{H}_{Q,W} \tilde{\boxtimes}_+ \mathrm{H}_{Q,W} := \bigoplus_{\mathbf{d} \in \mathbb{Z}_{\geqslant 0}^{Q_0}} \bigoplus_{\mathbf{d}=\mathbf{e}+\mathbf{f}} \mathrm{H}_{Q,W,\mathbf{e}} \otimes \mathrm{H}_{Q,W,\mathbf{f}} \left[\prod_{i,j \in Q_0} \prod_{\substack{1 \leqslant s \leqslant \mathbf{e}_i \\ 1 \leqslant t \leqslant \mathbf{f}_j}} (y_{j,t} - x_{i,s})^{-1} \right],$$

where:

$$\mathrm{H}_{\mathrm{GL}_\mathbf{e}}(\mathrm{pt}) = \mathbb{Q}\left[x_{i,k},\ {}^{i \in Q_0}_{1 \leqslant k \leqslant \mathbf{e}_i}\right]^{\mathcal{S}_\mathbf{e}} \ ; \ \mathrm{H}_{\mathrm{GL}_\mathbf{f}}(\mathrm{pt}) = \mathbb{Q}\left[y_{i,k},\ {}^{i \in Q_0}_{1 \leqslant k \leqslant \mathbf{f}_i}\right]^{\mathcal{S}_\mathbf{f}}.$$

In order to obtain a bialgebra, we need to multiply elements in $\mathrm{H}_{Q,W}\tilde{\boxtimes}_{+}\mathrm{H}_{Q,W}$ i.e. we need a morphism:

$$\mathrm{m}^2 : (\mathrm{H}_{Q,W}\tilde{\boxtimes}_{+}\mathrm{H}_{Q,W}) \boxtimes_{+} (\mathrm{H}_{Q,W}\tilde{\boxtimes}_{+}\mathrm{H}_{Q,W}) \to (\mathrm{H}_{Q,W}\tilde{\boxtimes}_{+}\mathrm{H}_{Q,W}).$$

On elements $a \otimes b \otimes c \otimes d$ in a suitable localization of $\mathrm{H}_{Q,W,\mathbf{d}_1} \otimes \mathrm{H}_{Q,W,\mathbf{d}_2} \otimes \mathrm{H}_{Q,W,\mathbf{d}_3} \otimes \mathrm{H}_{Q,W,\mathbf{d}_4}$, m^2 is defined as:

$$\mathrm{m}^2(a \otimes b \otimes c \otimes d) := (-1)^{\deg(b)\cdot\deg(c)+\chi(\mathbf{d}_2,\mathbf{d}_2)\chi(\mathbf{d}_3,\mathbf{d}_3)+\chi(\mathbf{d}_2,\mathbf{d}_3)}\mathrm{m}(a \otimes c) \otimes \mathrm{m}(b \otimes d).$$

It turns out that $\mathrm{m}^2(a \otimes b \otimes c \otimes d)$ lives in the required localization of $\mathrm{H}_{Q,W,\mathbf{d}_1+\mathbf{d}_3} \otimes \mathrm{H}_{Q,W,\mathbf{d}_2+\mathbf{d}_4}$ (see [2, §5.1.] for details).

4.2 Construction of a coproduct

Let us now describe the following component of the localised coproduct Δ ($\mathbf{d} = \mathbf{e} + \mathbf{f}$):

$$\Delta_{\mathbf{e},\mathbf{f}} : \mathrm{H}_{Q,W,\mathbf{d}} \to \mathrm{H}_{Q,W,\mathbf{e}} \otimes \mathrm{H}_{Q,W,\mathbf{f}} \left[\prod_{i,j \in Q_0} \prod_{\substack{1 \leqslant s \leqslant \mathbf{e}_i \\ 1 \leqslant t \leqslant \mathbf{f}_j}} (y_{j,t} - x_{i,s})^{-1} \right].$$

Define the following symmetric polynomials:

$$E_{1,\mathbf{e},\mathbf{f}} := \prod_{\substack{a \in Q_1 \\ a: i \to j}} \prod_{\substack{1 \leqslant s \leqslant \mathbf{e}_i \\ 1 \leqslant t \leqslant \mathbf{f}_j}} (x_{i,s} - y_{j,t}),$$

$$E_{0,\mathbf{e},\mathbf{f}} := \prod_{i \in Q_0} \prod_{\substack{1 \leqslant s \leqslant \mathbf{e}_i \\ 1 \leqslant t \leqslant \mathbf{f}_i}} (x_{i,s} - y_{i,t}).$$

Then $\Delta_{\mathbf{e},\mathbf{f}}$ is defined as the following composition of morphisms, shifted by $\chi(\mathbf{d},\mathbf{d})$ and followed by multiplication by $E_{1,\mathbf{e},\mathbf{f}}^{-1} \cdot E_{0,\mathbf{e},\mathbf{f}}$:

$$\begin{aligned}
\mathrm{H}_{Q,W,\mathbf{d}}[-\chi(\mathbf{d},\mathbf{d})] &\to \mathrm{H}^{\bullet}_c(X_{\mathbf{d}}/\mathrm{GL}_{\mathbf{e},\mathbf{f}}, \phi_{\mathrm{Tr}(W)}\underline{\mathbb{Q}}[-1])^{\vee}[-2\mathbf{e}\cdot\mathbf{f}] && \text{(2.5.iv.)}\\
&\to \mathrm{H}^{\bullet}_c(\mathfrak{M}_{\mathbf{e},\mathbf{f}}, \phi_{\mathrm{Tr}(W)}\underline{\mathbb{Q}}[-1])^{\vee}[-2\chi(\mathbf{e},\mathbf{f})] && \text{(2.5.ii.)}\\
&\simeq \mathrm{H}^{\bullet}_c(X_{\mathbf{e},\mathbf{f}}/(\mathrm{GL}_{\mathbf{e}} \times \mathrm{GL}_{\mathbf{f}}), \phi_{\mathrm{Tr}(W)}\underline{\mathbb{Q}}[-1])^{\vee}[-2\chi(\mathbf{e},\mathbf{f}) - 2\mathbf{e}\cdot\mathbf{f}] && \text{(2.5.iii.)}\\
&\simeq \mathrm{H}^{\bullet}_c(\mathfrak{M}_{\mathbf{e}} \times \mathfrak{M}_{\mathbf{f}}, \phi_{\mathrm{Tr}(W)_{\mathbf{e}} \boxplus \mathrm{Tr}(W)_{\mathbf{f}}}\underline{\mathbb{Q}}[-1])^{\vee}[-2\chi(\mathbf{e},\mathbf{f}) - 2\chi(\mathbf{f},\mathbf{e})] && \text{(2.5.i.)}\\
&\simeq \mathrm{H}^{\bullet}_c(\mathfrak{M}_{\mathbf{e}}, \phi_{\mathrm{Tr}(W)}\underline{\mathbb{Q}}[-1])^{\vee} \otimes \mathrm{H}^{\bullet}_c(\mathfrak{M}_{\mathbf{f}}, \phi_{\mathrm{Tr}(W)}\underline{\mathbb{Q}}[-1])^{\vee}[-2\chi(\mathbf{e},\mathbf{f}) - 2\chi(\mathbf{f},\mathbf{e})]. && \text{(Thom-Sebastiani)}
\end{aligned}$$

Note that this map is degree preserving, since $E_{1,\mathbf{e},\mathbf{f}}^{-1} \cdot E_{0,\mathbf{e},\mathbf{f}}$ has cohomological degree $2\chi(\mathbf{e},\mathbf{f})$. The main theorem of [2] is then:

Theorem 4.1. *[2, Thm. 5.13.] The following diagram commutes. In other words, $\mathrm{H}_{Q,W}$ is a Q-localised bialgebra.*

$$\begin{array}{ccc} \mathrm{H}_{Q,W}\boxtimes_+\mathrm{H}_{Q,W} & \xrightarrow{\Delta\boxtimes_+\Delta} & (\mathrm{H}_{Q,W}\tilde{\boxtimes}_+\mathrm{H}_{Q,W})\boxtimes_+(\mathrm{H}_{Q,W}\tilde{\boxtimes}_+\mathrm{H}_{Q,W}) \\ \downarrow{\scriptstyle \mathrm{m}} & & \downarrow{\scriptstyle \mathrm{m}^2} \\ \mathrm{H}_{Q,W} & \xrightarrow{\qquad\quad\Delta\qquad\quad} & \mathrm{H}_{Q,W}\tilde{\boxtimes}_+\mathrm{H}_{Q,W}. \end{array}$$

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