COHA OF A PREPROJECTIVE ALGEBRA AND SHUFFLE ALGEBRAS

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1. Preprojective algebra of a quiver

Let Q be a quiver with vertex set Q_0 and edge set Q_1. We allow Q to have multiple edges and loops. Given a dimension vector $\mathbf{d} = (d_i)_{i\in Q_0} \in \mathbb{Z}_{\geq 0}^{|Q_0|}$, we let $\mathrm{Rep}(Q, \mathbf{d})$ denote the affine space of representations of Q of dimension $\mathbf{d}$.

Let $\overline{Q}$ denote the doubled quiver of Q; the quiver $\overline{Q}$ has vertex set Q_0, and edge set $\{x, x^*|x \in Q_1\}$, where x^* points in the opposite direction of x. Given a dimension vector $\mathbf{d}$ for Q (and $\overline{Q}$), there is a canonical isomorphism $\mathrm{Rep}(\overline{Q}, \mathbf{d}) \cong T^*\mathrm{Rep}(Q, \mathbf{d})$ induced by the trace pairing.

The group $GL_{\mathbf{d}} := \prod_{i\in Q_0} GL_{d_i}$ acts on $\mathrm{Rep}(Q, d)$ by conjugation at each vertex. This action induces a Hamiltonian action on the symplectic vector space $\mathrm{Rep}(\overline{Q}, \mathbf{d})$. Under the identification $\mathfrak{gl}_{\mathbf{d}}^\vee \cong \mathfrak{gl}_{\mathbf{d}}$ given by the Killing form, this action has moment map $\mu_{\mathbf{d}} : \mathrm{Rep}(\overline{Q}, \mathbf{d}) \to \mathfrak{gl}_{\mathbf{d}}$ given by

$$(x, x^*)_{x\in Q_1} \mapsto \sum_{x\in Q_1} [x, x^*].$$

Let $\mathbb{C}Q$ denote the path algebra of Q. The *preprojective algebra* Π_Q of Q is defined to be the quotient

$$\mathbb{C}\overline{Q} \Big/ \sum_{x\in Q_1} [x, x^*].$$

In other words, Π_Q-modules are those $\mathbb{C}\overline{Q}$-modules which are annihilated by $\mu_{\mathbf{d}}^{-1}(0)$.

A key feature of the algebras Π_Q is the following:

Proposition 1.1. [SV20, e.g. Proposition 3.1] *For Q not of Dynkin type, the algebra Π_Q is 2-Calabi-Yau.*

This result can be extended to the case when Q is Dynkin by replacing Π_Q with a dg analog; see [DHM22, 2.2] or [K20, 0.2.15]. Namely, let $\tilde{Q}$ denote the triple quiver obtained from $\overline{Q}$ by adding a new loop t_i at each $i \in Q_0$ and equip $\mathbb{C}\tilde{Q}$ with the grading such that t_i have degree -1 and paths in $\overline{Q}$ have degree 0. If d is the unique differential on $\mathbb{C}\tilde{Q}$ sending $t_i \mapsto e_i \sum_{x\in Q_1}[x, x^*]e_i$ where $e_i \in \mathbb{C}Q$ is the constant path at i, then the differential graded algebra $(\mathbb{C}\tilde{Q}, d)$ is differential-graded 2-Calabi-Yau. The dga $(\mathbb{C}\tilde{Q}, d)$ has zeroth cohomology Π_Q; for Q non-Dynkin, the map $(\mathbb{C}\tilde{Q}, d) \to \Pi_Q$ is a quasi-isomorphism.

2. Cohomological Hall algebra of a preprojective algebra

2.1. **Definition.** Our aim is to equip the equivariant homology of the moduli stack

$$\sqcup_d \, \mu_{\mathbf{d}}^{-1}(0)/GL_{\mathbf{d}}$$

of Π_Q-representations with the structure of a cohomological Hall algebra. We follow [YZ18, §4-5] (see also [YZ20, §3.1-3.2]), which extends to arbitrary Q an argument developed for the Jordan quiver in [SV13a, SV13b].

Let $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{Z}_{\geq 0}^{|Q_0|}$ be two dimension vectors corresponding to $|Q_0|$-tuples V_1 and V_2 of vector spaces and set $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$. Let $G = GL_{\mathbf{d}}$, P be the parabolic subgroup $\{g \in G | g(V_1) \subset V_1\}$ and L be the Levi subgroup $GL_{\mathbf{d}_1} \times GL_{\mathbf{d}_2}$. Let $\mathfrak{g}, \mathfrak{p}$ and $\mathfrak{l}$ be the respective Lie algebras. We set

$$\mathrm{Rep}(Q)_{\mathbf{d}_1,\mathbf{d}_2} = \{x \in \mathrm{Rep}(Q,\mathbf{d}) | x(V_1) \subset V_1\},$$

and define $\mathrm{Rep}(\overline{Q})_{\mathbf{d}_1,\mathbf{d}_2}$ analogously.

We previously saw that the stack of representations of a quiver can be equipped with a COHA structure using pull-push along the following correspondence of G-varieties:

$$G \times_P \big(\mathrm{Rep}(Q,\mathbf{d}_1) \times \mathrm{Rep}(Q,\mathbf{d}_2)\big) \leftarrow G \times_P \mathrm{Rep}(Q)_{\mathbf{d}_1,\mathbf{d}_2} \to \mathrm{Rep}(Q,\mathbf{d}), \tag{2.1}$$

where the left arrow is projection and the right arrow is G-action.

For the stack of preprojective representations, a first attempt could be to run the same procedure for the analogous correspondence

$$G \times_P \left(\mu_{\mathbf{d}_1}^{-1}(0) \times \mu_{\mathbf{d}_2}^{-1}(0)\right) \xleftarrow{\overline{g}} G \times_P \left(\mu_{\mathbf{d}}^{-1}(0) \cap \mathrm{Rep}(\overline{Q})_{\mathbf{d}_1,\mathbf{d}_2}\right) \xrightarrow{\overline{h}} \mu_{\mathbf{d}}^{-1}(0). \tag{2.2}$$

The map $\overline{h}$ is proper, but the map $\overline{g}$ need not be a locally complete intersection. So, the pullback $\overline{g}^*$ is not well-defined.

The fix is to include (2.2) in the correspondence (2.1) for $\overline{Q}$ with the left term slightly enlarged. Let $\pi : \mathfrak{p} \to \mathfrak{l}$ denote the projection and define

$$\overline{Z}_{\mathbf{d}_1,\mathbf{d}_2} = \Big\{(a, \overline{x}_1, \overline{x}_2) \in \mathfrak{p} \times \mathrm{Rep}(\overline{Q},\mathbf{d}_1) \times \mathrm{Rep}(\overline{Q},\mathbf{d}_2) \mid \pi(a) = \big(\mu_{\mathbf{d}_1}(\overline{x}_1), \mu_{\mathbf{d}_2}(\overline{x}_2)\big)\Big\}.$$

Consider the diagram

$$\begin{array}{ccccccc}
G \times_P \left(\mu_{\mathbf{d}_1}^{-1}(0) \times \mu_{\mathbf{d}_2}^{-1}(0)\right) & & \xleftarrow{\quad\overline{g}\quad} & G \times_P \left(\mu_{\mathbf{d}}^{-1}(0) \cap \mathrm{Rep}(\overline{Q})_{\mathbf{d}_1,\mathbf{d}_2}\right) & \xrightarrow{\overline{h}} & \mu_{\mathbf{d}}^{-1}(0) \\
\big\downarrow & & & \big\downarrow & & \big\downarrow \\
G \times_P \big(\mathrm{Rep}(\overline{Q},\mathbf{d}_1) \times \mathrm{Rep}(\overline{Q},\mathbf{d}_2)\big) & \hookrightarrow G \times_P \overline{Z}_{\mathbf{d}_1,\mathbf{d}_2} & \xleftarrow{\quad g\quad} & G \times_P \mathrm{Rep}(\overline{Q})_{\mathbf{d}_1,\mathbf{d}_2} & \xrightarrow{\quad h\quad} & \mathrm{Rep}(\overline{Q},\mathbf{d})
\end{array} \tag{2.3}$$

of G-varieties. Then the left-hand square is Cartesian and g is a map between smooth varieties and is therefore l.c.i.. Also, note that $\overline{h}$ is proper.

So, let A_* denote any oriented Borel-Moore homology theory (examples include locally finite singular homology and Chow with rational coefficients). Then, there exists a refined

Gysin pullback map

$$g^\sharp : A_G\Big(G \times_P \big(\mu_{\mathbf{d}_1}^{-1}(0) \times \mu_{\mathbf{d}_2}^{-1}(0)\big)\Big) \to A_G\Big(G \times_P \big(\mu_{\mathbf{d}}^{-1}(0) \cap \mathrm{Rep}(\overline{Q})_{\mathbf{d}_1,\mathbf{d}_2}\big)\Big).$$

The constructions in the previous paragraph can be equipped with the following helpful extra equivariance. Let $T = \mathrm{diag}(t_1, t_2)$ act on $\mathrm{Rep}(\overline{Q}, \mathbf{d})$ such that all paths x have weight t_1 and all paths x^* have weight t_2. (This assumption can be slightly weakened at the cost of complicating the explicit shuffle formulas; see [YZ18, Assumption 3.1] and [SV20, §3.3].) Then, the diagram (2.3) is $G \times T$-equivariant.

Set $\mathcal{P}(Q) = \oplus_{\mathbf{d}} \mathcal{P}_{\mathbf{d}}(Q) = \oplus_{\mathbf{d}} A_{GL_{\mathbf{d}} \times T}(\mu_{\mathbf{d}}^{-1}(0))$. The COHA product $\mathcal{P}_{\mathbf{d}_1} \otimes_{A(\mathrm{pt})[[t_1,t_2]]} \mathcal{P}_{\mathbf{d}_2} \to \mathcal{P}_{\mathbf{d}_1+\mathbf{d}_2}$ is defined as the composition of the following maps.

- The Künneth homomorphism $P_{\mathbf{d}_1} \otimes_{A(\mathrm{pt})[[t_1,t_2]]} P_{\mathbf{d}_2} \to A_{L\times T}(\mu_{\mathbf{d}_1}^{-1}(0) \times \mu_{\mathbf{d}_2}^{-1}(0))$
- The isomorphism $A_{L\times T}(\mu_{\mathbf{d}_1}^{-1}(0) \times \mu_{\mathbf{d}_2}^{-1}(0)) \cong A_{G\times T}\Big(G \times_P \big(\mu_{\mathbf{d}_1}^{-1}(0) \times \mu_{\mathbf{d}_2}^{-1}(0)\big)\Big)$ induced by the homotopy equivalence $P \twoheadrightarrow L$.
- The refined Gysin pullback

$$g^\sharp : A_{G\times T}\Big(G \times_P \big(\mu_{\mathbf{d}_1}^{-1}(0) \times \mu_{\mathbf{d}_2}^{-1}(0)\big)\Big) \to A_{G\times T}\Big(G \times_P \big(\mu_{\mathbf{d}}^{-1}(0) \cap \mathrm{Rep}(\overline{Q})_{\mathbf{d}_1,\mathbf{d}_2}\big)\Big).$$

- The proper pushforward $\overline{h}_* : A_{G\times T}\Big(G \times_P \big(\mu_{\mathbf{d}}^{-1}(0) \cap \mathrm{Rep}(\overline{Q})_{\mathbf{d}_1,\mathbf{d}_2}\big)\Big) \to P_{\mathbf{d}}$.

Theorem 2.1. [SV13a, Proposition 7.5],[YZ18, Theorem 4.1] *Equipped with the above product, the homology $\mathcal{P}(Q)$ carries the structure of a $\mathbb{Z}_{\geq 0}^{|Q_0|}$-graded associative algebra over $A_T(pt)$.*

The associative algebra $\mathcal{P}(Q)$ is called the preprojective COHA.

2.2. **Properties.** The correspondence (2.3) is a special case of a robust Lagrangian correspondence formalism that can be applied to define more examples of COHA and establish more properties. For example, such a correspondence produces COHA actions on homology of Nakajima quiver varieties.

Given a quiver Q, let Q^f denote the framed quiver consisting of Q along with a new set Q_0' of $|Q_0|$ vertices and a new set of $|Q_0|$ edges $\{y_i\}_{i\in Q_0}$ such that the new edges induce a bijection $Q_0' \to Q_0$. Given dimension vectors $\mathbf{d}, \mathbf{w}$ corresponding to $|Q_0|$-tuples V, W of vector spaces, let $\mathrm{Rep}(\overline{Q^f}, \mathbf{d}, \mathbf{w})$ denote the set of representations of the doubled framed quiver $\overline{Q^f}$ with dimension vector $\mathbf{d}$ at Q_0 and $\mathbf{w}$ at Q_0' and let $\mu_{\mathbf{d},\mathbf{w}} : \mathrm{Rep}(\overline{Q^f}, \mathbf{d}, \mathbf{w}) \to \mathfrak{gl}_{\mathbf{d}}$ be the associated moment map.

For our purposes, given some stability condition $\theta : GL_{\mathbf{d}} \to \mathbb{G}_m$, the Nakajima quiver variety is defined as

$$\mathcal{M}_\theta(Q, \mathbf{d}, \mathbf{w}) := \mu_{\mathbf{d},\mathbf{w}}^{-1}(0)^{\theta^{\mathrm{ss}}} // GL_{\mathbf{d}},$$

where here and elsewhere the superscript θ^{ss} denotes the subset of θ-semistable representations.

Now, given dimension vectors $\mathbf{d}_1, \mathbf{d}_2, \mathbf{w}$ corresponding to tuples of vectors spaces V_1, V_2 and W with $\mathbf{d}_1 + \mathbf{d}_2 = \mathbf{d}$, we introduce the following correspondence between $\overline{Q}^f$-representations and $\overline{Q}$ representations. Set

$$\operatorname{Rep}(\overline{Q^f}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{w}) := \Big\{(x, x^*), (y_i, y_i^*) \in \operatorname{Rep}(\overline{Q^f}, \mathbf{d}, \mathbf{w}) | x(V_1) \subset V_1, x^*(V_1) \subset V_1, y_i(W_i) \subset V_1\Big\}.$$

Let $\theta_+ : GL_{\mathbf{d}} \to \mathbb{G}_m$ denote the stability condition sending $(g_i)_{i \in Q_0} \mapsto \prod_{i \in Q_0} \det(g_i)^{-1}$.

Then, set

$$\overline{Z^f}_{\mathbf{d}_1, \mathbf{d}_2, \mathbf{w}} = \Big\{a, \overline{x_1}, \overline{x_2} \in \mathfrak{p} \times \operatorname{Rep}(\overline{Q_f}(\mathbf{d}_1, \mathbf{w}))^{\theta_+^{\mathrm{ss}}} \times \operatorname{Rep}(\overline{Q}, \mathbf{d}_2) \Big| \pi(a) = \big(\mu_{\mathbf{d}_1}(\overline{x}_1), \mu_{\mathbf{d}_2}(\overline{x}_2)\big)\Big\}.$$

One then has the following analog of (2.3).

$$\begin{array}{ccccc}
G \times_P \big(\mu^{-1}_{\mathbf{d}_1, \mathbf{w}}(0)^{\mathrm{ss}} \times \mu^{-1}_{\mathbf{d}_2}(0)\big) & \xleftarrow{\overline{g}'} & G \times_P \big(\mu^{-1}_{\mathbf{d}, \mathbf{w}}(0)^{\mathrm{ss}} \cap \operatorname{Rep}(\overline{Q^f})_{\mathbf{d}_1, \mathbf{d}_2, \mathbf{w}}\big) & \xrightarrow{\overline{h}'} & \mu_{\mathbf{d}}^{-1}(0)^{\mathrm{ss}} \\
\downarrow & & \downarrow & & \downarrow \\
G \times_P \overline{Z^f}_{\mathbf{d}_1, \mathbf{d}_2, \mathbf{w}} & \xleftarrow{g'} & G \times_P \operatorname{Rep}(\overline{Q^f})^{\mathrm{ss}}_{\mathbf{d}_1, \mathbf{d}_2, \mathbf{w}} & \xrightarrow{h'} & \operatorname{Rep}(\overline{Q^f}, \mathbf{d}, \mathbf{w})^{\mathrm{ss}}
\end{array} \tag{2.4}$$

where all semistability is taken with respect to θ_+. As for (2.3), the left square is Cartesian and g' is l.c.i..

Following the same procedure as in the previous section, the correspondences (2.4) yield multiplication maps

$$\mathcal{P}_{\mathbf{d}_1} \otimes A_{GL_{\mathbf{d}_2} \times T \times GL_{\mathbf{w}}}\big(\mathcal{M}_{\theta_+}(Q, \mathbf{d}_2, \mathbf{w})\big) \to A_{GL_{\mathbf{d}_1 + \mathbf{d}_2} \times T \times GL_{\mathbf{w}}}\big(\mathcal{M}_{\theta_+}(Q, \mathbf{d}_1 + \mathbf{d}_2, \mathbf{w})\big).$$

An argument similar to that used to prove associativity of the COHA multiplication implies the following.

Theorem 2.2. [SV13a, Proposition 7.9],[YZ18, Theorem 5.4] *Fix $\mathbf{w}$. Then, the sum*

$$\oplus_{\mathbf{d}} A_{GL_{\mathbf{d}} \times T}(\mathcal{M}_{\theta_+}(Q, \mathbf{d}, \mathbf{w}))$$

carries the structure of a $\mathcal{P}(Q)$-representation.

It is also fruitful to study COHA structures for Serre subcategories of $\Pi_Q - \operatorname{Rep}$. For example, [SV20, DHM22] construct and use COHA for subcategories of preprojective representations satisfying certain semi-nilpotency conditions. Namely, a quiver representation is said to be semi-nilpotent if it admits a flag F_j such that $x(F_j) \subset F_{j-1}$ and $x^*(F_j) \subset F_j$, and strongly semi-nilpotent if there exists such a flag for which each quotient L_j/L_{j-1} is nonzero on only one vertex. Moduli of such representations comprise Lagrangian subvarieties of $\operatorname{Rep}(\overline{Q})$. In [SV20], the resulting COHA are shown to act on the Borel-Moore homology of Nakajima quiver varieties for generic stability conditions θ. We remark that in [SV20], the COHA is defined to be a certain extension of the algebra on the stack of semi-nilpotent representations by a commutative algebra, given by multiplication by Chern classes of tautological bundles.

3. Embedding in a shuffle algebra

The Hall products given by the two correspondences in (2.3) are compatible in the following sense.

Proposition 3.1. [SV13b, Proposition 4.6],[YZ18, Theorem 5.3] *The $GL_{\mathbf{d}} \times T$-equivariant closed embeddings $\iota : \mu_{\mathbf{d}}^{-1}(0) \hookrightarrow \operatorname{Rep}(\overline{Q}, \mathbf{d})$ induce homomorphisms*

$$A_{GL_{\mathbf{d}} \times T}(\mu_{\mathbf{d}}^{-1}(0)) \to A_{GL_{\mathbf{d}} \times T} \operatorname{Rep}(\overline{Q}, \mathbf{d}) \tag{3.1}$$

which intertwine the Hall multiplication.

The Proposition is a consequence of the following special case of the proper base change property for the refined Gysin pullback ([LM07, Theorem 6.6.2(a)]). Suppose the diagram

$$\begin{array}{ccc} W' & \xrightarrow{\overline{f}} & X' \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} \\ W & \xrightarrow{f} & X \end{array}$$

is Cartesian, that f is l.c.i. and that j is proper. Then $i_* f^\sharp = f^* j_*$.

In the previous talk, we learned that that Hall product on the equivariant cohomology of the affine spaces $\operatorname{Rep}(\overline{Q}, \mathbf{d})$ can be identified with the shuffle product. After incorporating the equivariant parameters of the the torus T, the Hall product under the identification

$$\oplus_{\mathbf{d}} H^*_{GL_{\mathbf{d}} \times T}\big(\operatorname{Rep}(\overline{Q}, \mathbf{d})\big) \cong \mathbb{Q}[t_1, t_2][z_s^{(i)} \mid i \in Q_0, 1 \le s \le d_i]^{\prod_i \operatorname{Sym}(d_i)},$$

takes the following form. Given

$$f_k \in H^*_{GL_{\mathbf{d}_k} \times T}\big(\operatorname{Rep}(\overline{Q}, \mathbf{d}_k)\big), \ k = 1, 2,$$

one has

$$f_1(z') * f_2(z'') = \sum_{\sigma \in \prod_i \operatorname{Sh}((d_1)_i, (d_2)_i)} \sigma(f_1 \cdot f_2 \cdot T_1 \cdot T_2),$$

where

$$T_1 = \prod_{i \in Q_0} \prod_{s=1}^{(d_1)_i} \prod_{t=1}^{(d_2)_i} \frac{-z_t^{(i)''} + z_s^{(i)'} + t_1 + t_2}{z_t^{(i)''} - z_s^{(i)'}}$$

$$T_2 = \prod_{i,j \in Q_0} \prod_{s=1}^{(d_1)_i} \prod_{t=1}^{(d_2)_j} \big(z_t^{(j)''} - z_s^{(i)'} + t_1\big)^{a_{ij}} \big(z_t^{(j)''} - z_s^{(i)'} + t_2\big)^{a_{ji}},$$

and a_{ij} denotes the number of edges from i to j in Q. The same formula holds for an arbitrary oriented cohomology theory after replacing addition with the associated formal group law; see [YZ18, §3.1]. Let $\mathcal{SH}$ denote $\sqcup_{\mathbf{d}} A_{GL_{\mathbf{d}} \times T}\big(\operatorname{Rep}(\overline{Q}, \mathbf{d})\big)$ equipped with the shuffle product.

For example, suppose Q is the Jordan quiver. Working in equivariant cohomology with complex coefficients, the degree $d = 1$ piece of the map

$$\sqcup_d \ H^*_{GL_d \times T}(\mu_1^{-1}(0)) \to \sqcup_d \ \mathbb{C}[t_1, t_2][z_1, ..., z_d]^{\mathrm{Sym}(d)}$$

takes the form

$$H^*_{(\mathbb{G}_m)_z \times T}(\mu_1^{-1}(0)) \ni z^m[\mu_1^{-1}(0)] \mapsto z_1^m \in \mathbb{C}[t_1, t_2][z];$$

see [SV13b, Theorem 4.7].

More generally, the spherical subalgebras $\mathcal{P}^{\mathrm{s}} \subset \mathcal{P}$ and $\mathcal{SH}^{\mathrm{s}} \subset \mathcal{SH}$ are defined to be the algebras generated by $\mathcal{P}_{\mathbf{d}}$ and $A_{GL_{\mathbf{d}} \times T}(\mathrm{Rep}(\overline{Q}, \mathbf{d}))$ with $|\mathbf{d}| = 1$, respectively. As (3.1) are isomorphisms for $\mathbf{d} = 1$, the induced map $\mathcal{P}^{\mathrm{s}} \to \mathcal{SH}^{\mathrm{s}}$ is surjective.

In general, the maps (3.1) are injective after localization but are not known to be injective in general. When the stack of preprojective representations is replaced by either the stack of semi-nilpotent representations or the stack of strongly semi-nilpotent representations considered in [SV20, DHM22], the injectivity of (3.1) is proved for Borel-Moore homology in [SV20, Proposition 4.6] using results of [D16]. In the case of strongly semi-nilpotent representations, [SV20, Theorem 5.18] furnishes a generating set for (an extension of) the associated COHA. This COHA can therefore be characterized as a shuffle algebra on certain explicit generators.

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