COHOMOLOGICAL HALL ALGEBRA OF A QUIVER AND SHUFFLE ALGEBRAS

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In Section 1 we associate to a quiver a moduli stack $M = \cup_\gamma M_\gamma / G_\gamma$ of representations. In Section 2 we equip the cohomology ring of M with a new product structure to obtain a cohomological Hall algebra. An explicit formula for the product is discussed in Section 3. Examples are discussed in Section 4. The main reference is [KS11, Section 1].

1. Moduli of quiver representations

1.0.1. *Quivers.* A *quiver* is a directed graph where we allow multiple edges and loops. See Figure 1 for examples. We write Q for a quiver; I for the set of vertices and $a_{i,j}$ for the number of edges from $i \in I$ to $j \in I$.

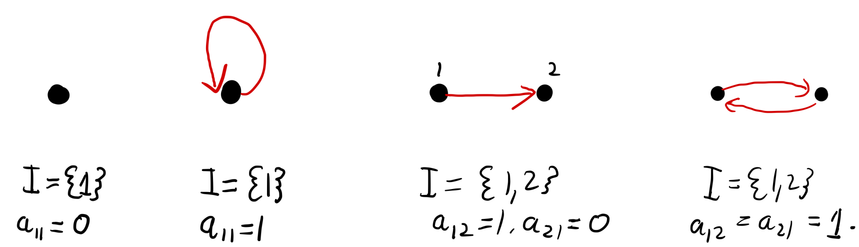

FIGURE 1. Examples of quivers.

1.0.2. *Quiver representations.* Fix now a positive integer assigned to each vertex of Q, called the *dimension vector* and denoted $\gamma = (\gamma^i)_{i \in I}$. A *representation* of Q of dimension γ is the data of a $\gamma^j \times \gamma^i$ complex matrix for every arrow from vertex i to vertex j. We identify two representations of Q if there is an element of $\prod_{i \in I_Q} \mathsf{GL}(\gamma^i, \mathbb{C})$ sending one representation to the other. Working with isomorphism classes of quiver representations amounts to forgetting a choice of basis for the vector space associated to each vertex.

1.0.3. *The space of representations of Q in complex vector spaces.* Fix Q, γ as in Section 1.0.1 and consider

$$M_\gamma = \prod_{i,j} \mathsf{Hom}(\mathbb{C}^{\gamma^i}, \mathbb{C}^{\gamma^j})^{a_{i,j}} = \prod_{i,j} \mathbb{C}^{a_{ij}\gamma^i\gamma^j}, \quad G_\gamma = \prod_{i \in I} \mathrm{GL}(\gamma_i, \mathbb{C}).$$

A point of M_γ is thus the data of a_{ij} matrices of dimension $\gamma^i \times \gamma^j$. The *stack of representations of Q with dimension vector γ* is the quotient stack $[M_\gamma / G_\gamma]$.

Remark 1.1. The vector γ is specifying a connected component of the moduli space of quiver representations. Each connected component is the global quotient stack of an affine space.

1.1. **Cohomology of the quotient stack.** The cohomology of $[M_\gamma / G_\gamma]$ is (defined to be) the G_γ equivariant cohomology of M_γ which we write as $H^\bullet_{G_\gamma}(M_\gamma)$. We now recall properties of equivariant cohomology.

1.1.1. *Equivariant cohomology.* If a group G acts freely on a scheme Y define G equivariant cohomology of Y

$$H^\bullet_G(Y) = H^\bullet(Y/G).$$

If G does not act freely on Y one instead constructs a space $\mathsf{E}G$ with cohomology isomorphic to the cohomology of a point and a free action of G on $\mathsf{E}G$. The Kunneth isomorphism identifies the cohomology of $\mathsf{E}G \times Y$ with the cohomology of Y. We are now free to define

$$H^\bullet_G(Y) = H^\bullet_G((Y \times \mathsf{E}G)/G).$$

1.1.2. *Models of the classifying space of G.* For us all groups G will be subgroups of $GL(n, \mathbb{C})$.

Example 1.2. Whenever G is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ we can take $\mathsf{E}G$ to be the space of ordered tuples of n linearly independent sequences of complex numbers. The group $G = \mathsf{GL}(n, \mathbb{C})$ acts freely on $\mathsf{E}G$ and the quotient is the *infinite Grassmannian*

$$\mathsf{Gr}(d, \mathbb{C}^\infty) = \varinjlim(\mathsf{Gr}(d, \mathbb{C}^n)) \text{ equipped with universal family } \mathsf{E}G.$$

The cohomology of $\mathsf{E}G/G = \mathsf{B}G$ is a polynomial ring in n variables $\mathbb{Z}[s_1, ..., s_n]$.

We call any weakly contractible space on which G acts freely a *model* of $\mathsf{E}G$ and the quotient of any model of $\mathsf{E}G$ by G is called a *model* of $\mathsf{B}G$.

Remark 1.3. Equivariant cohomology is independent of the model of $\mathsf{E}G$. Thus if K is a subgroup of G then $\mathsf{E}G$ with induced K action is a model of $\mathsf{E}K$ and $\mathsf{E}G/K$ is a model of $\mathsf{B}K$. In this way we induce a morphism

$$H_G(Y) \to H_K(Y).$$

Example 1.4. (The $\mathrm{GL}(n, \mathbb{C})$ equivariant cohomology of a point.) Associated to the the diagonal subgroup

$$(\mathbb{C}^\star)^n \to \mathrm{GL}(n, \mathbb{C})$$

there is a pullback map on cohomology

$$H^\bullet\left(\mathrm{BGL}(n, \mathbb{C})\right) \to H^\bullet\left(B(\mathbb{C}^\star)^n\right).$$

Thinking of the torus equivariant cohomology of a point as polynomials in formal variables $x_1, ..., x_n$, this map sends s_i to the i^{th} symmetric polynomial in the x_i variables.

2. Hall algebra from quiver representations

Define a $\mathbb{Z}^I_{\geq 0}$ graded abelian group

$$\mathcal{H} = \bigoplus_{\gamma \in \mathbb{Z}^I_{\geq 0}} H_\gamma \text{ where } H_\gamma = \bigoplus_{n \in \mathbb{Z}} H^n([M_\gamma/G_\gamma]).$$

This is simply the cohomology group of $\cup_\gamma M_\gamma$, but we have forgotten the ring structure.

Remark 2.1. The multiplication defined in this section is associative and preserves the $\mathbb{Z}^I_{\geq 0}$ grading but does not respect the cohomological grading. The ordinary cohomology unit makes our multiplication unital.

2.1. **Multiplication.** The multiplication map is the data of a morphism of rings

$$m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \quad \text{which we express as a sum } m = \sum_{\gamma_1, \gamma_2 \in \mathbb{Z}^I_{\geq 0}} m_{\gamma_1, \gamma_2}.$$

Our task now is to specify m_{γ_1, γ_2}.

Definition 2.2. The *cohomological Hall algebra* associated to the quiver Q is the algebra obtained by equipping the abelian group $\mathcal{H}$ with multiplication m.

2.1.1. *Notation.* Pick $\gamma_1, \gamma_2 \in \mathbb{Z}^I_{\geq 0}$ and denote $\gamma = \gamma_1 + \gamma_2$. Write M_{γ_1,γ_2} for the closed affine subspace of M_γ containing the standard coordinate subspace of dimension $(\gamma_1^i)_{i \in I}$ as a subrepresentation. A point of M_{γ_1,γ_2} is thus for each of the $a_{i,j}$ arrows between vertices i and j, a matrix of dimension $\gamma^i \times \gamma^j$ which is block upper triangular, see Figure 2. Define G_{γ_1,γ_2} the subgroup of G_γ preserving $\mathbb{C}^{\gamma_1} \leq \mathbb{C}^\gamma$ (again think block upper triangular matrices).

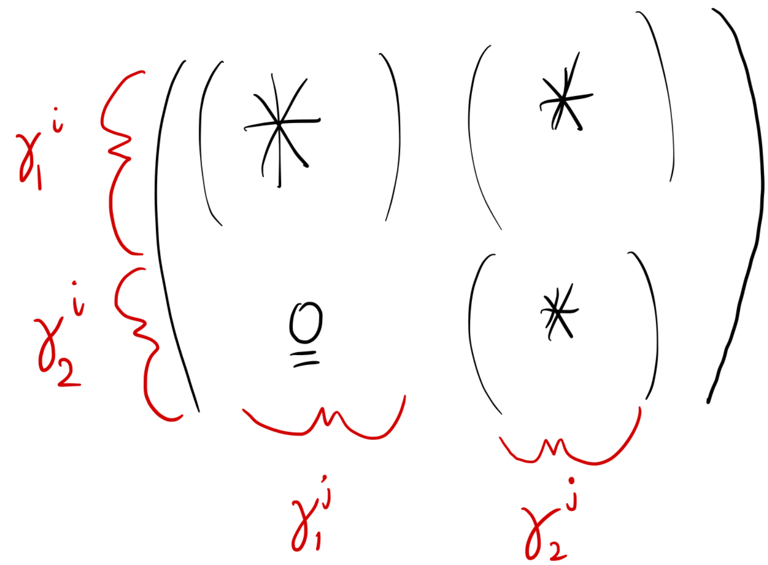

FIGURE 2. A point of M_{γ_1,γ_2} is specified by a matrix of the above form associated to each arrow of Q from vertex j to vertex i in I.

2.1.2. *Stacky definition for multiplication.* I find the following definition the easiest to process - the reader who prefers to avoid the language of stacks may skip to the next subsection. There are maps of stacks

$$M_{\gamma_1}/G_{\gamma_1} \times M_{\gamma_2}/G_{\gamma_2} \xleftarrow{g} M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2} \xrightarrow{h} M_\gamma/G_\gamma$$

Since h is a proper morphism of smooth Artin stacks there is an associated pushforward on cohomology $h_!$. We define m_{γ_1,γ_2} as the composition

$$h_! \circ g^\star : \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2} \to \mathcal{H}_\gamma.$$

2.1.3. *Rephrasing without stacks.* We rephrase the definition from Section 2.1.2 without mentioning stacks. Consider the map

$$\mathrm{Gr}_{\gamma_1,\gamma} := G_\gamma \times_{G_{\gamma_1,\gamma_2}} M_{\gamma_1,\gamma_2} \xrightarrow{\pi} M_\gamma, \quad (g, m) \mapsto gm.$$

This proper map induces a pushforward in cohomology

$$\pi_* : H^\bullet_{G_\gamma}(\mathrm{Gr}_{\gamma_1,\gamma}) \to H^{\bullet - 2\chi_Q(\gamma_1,\gamma_2)}_{G_\gamma}(M_\gamma).$$

The product structure can then be characterised through the composition

$$H^\bullet_{\mathrm{G}_{\gamma_1}}(M_{\gamma_1}) \otimes H^\bullet_{\mathrm{G}_{\gamma_2}}(M_{\gamma_2}) \to H^\bullet_{G_\gamma}(\mathrm{Gr}_{\gamma_1,\gamma}) \xrightarrow{\pi_\star} H^{\bullet - 2\chi_Q(\gamma_1,\gamma_2)}_{G_\gamma}(M_\gamma)$$

Remark 2.3. Note the new multiplication does not respect cohomological grading. Instead it induces a shift in grading of

$$2\chi_Q(\gamma_1,\gamma_2) = \left(-\sum_{i,j\in I} a_{ij}\gamma_1^j\gamma_2^i\right) + \left(\sum_{i\in I}\gamma_1^i\gamma_2^i\right) = 2c_2 + 2c_1.$$

2.1.4. *Multiplication via equivariant cohomology.* We break our definition down in the language of equivariant cohomology. The multiplication map is the composition

$$H^\bullet_{G_{\gamma_1}}(M_{\gamma_1})\otimes H^\bullet_{G_{\gamma_2}}(M_{\gamma_2}) \xrightarrow{\otimes} H^\bullet_{G_{\gamma_1}\times G_{\gamma_2}}(M_{\gamma_1}\times M_{\gamma_2}) = H^\bullet_{G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}) \xrightarrow{(3)} H^{\bullet+2c_1}_{G_{\gamma_1,\gamma_2}}(M_\gamma) \xrightarrow{(4)} H^{\bullet+2c_1+2c_2}_{G_\gamma}(M_\gamma)$$

where we now explain each map in this composition.

(1) The first morphism is induced by the Kunneth map.
(2) The equality follows from the equivariant homotopy equivalence

$$M_{\gamma_1,\gamma_2}\to M_{\gamma_1}\times M_{\gamma_2} \quad G_{\gamma_1,\gamma_2}\to G_{\gamma_1}\times G_{\gamma_2}.$$

(3) The second arrow is pushforward from a closed submanifold

$$M_{\gamma_1,\gamma_2}\to M_\gamma.$$

(4) The final arrow is a map

$$H^{\bullet+2c_1}_{G_{\gamma_1,\gamma_2}}(M_\gamma) = H^{\bullet+2c_1}(M_\gamma\times \mathsf{E}G_\gamma/G_{\gamma_1,\gamma_2}) \to H^{\bullet+2c_1}(M_\gamma\times \mathsf{E}G_\gamma/G_\gamma) = H^{\bullet+2c_1+2c_2}_{G_\gamma}(M_\gamma)$$

defined by integrating along fibres for the $G_\gamma/G_{\gamma_1,\gamma_2}$ bundle defined by the quotient map

$$\mathsf{E}G_\gamma/G_{\gamma_1,\gamma_2}\to \mathsf{E}G_\gamma/G_\gamma.$$

Remark 2.4. Consider the quotient of $\mathrm{GL}(n+m)$ by the subgroup of block upper triangular matrices in which the bottom left $n\times m$ block is zero. This quotient is the Grassmannian $\mathrm{Gr}(n,n+m)$. Indeed the stabiliser of the action of G_{γ_1,γ_2} on G_γ is isomorphic to $\mathbb{A}^n$. We deduce

$$G_\gamma/G_{\gamma_1,\gamma_2} = \prod_{i\in I}\mathrm{Gr}(\gamma_1^i,\gamma^i).$$

3. Explicit formula for the product

The additive abelian group underlying $\mathcal{H}$ is the equivariant cohomology of a disjoint union of affine spaces. Affine space is homotopic to a point and thus Example 1.2 identifies this the underlying abelian group of $\mathcal{H}$ with a direct sum of polynomial rings.

Fix two dimension vectors γ_1,γ_2. Identify the abelian subgroupgroup $\mathcal{H}_{\gamma_1}$ with the underlying abelian group of the ring of symmetric polynomials in formal variables $x'_{i,\alpha}$ where i is an element of I. The index α then ranges over γ_1^i possible values. Similarly identify $\mathcal{H}_{\gamma_2}$ with the underlying abelian group of the ring of symmetric polynomials in formal variables $x''_{i,\alpha}$ where α ranges over γ_2^i values. The maximal torus of G_γ is the product of the maximal tori in G_{γ_1} and G_{γ_2} so we may identify H_γ with symmetric polynomials in the variables $x''_{i,\alpha}, x'_{i,\alpha}$.

After fixing γ_1 and $\gamma=\gamma_1+\gamma_2$ we can talk about the set of *shuffles*. A shuffle is the data for each $i\in I$ of a dimension γ_1^i coordinate subspace of $\mathbb{C}^{\gamma^i}$. There are $\prod_i \binom{\gamma^i}{\gamma_1^i}$ such shuffles.

Theorem 3.1. With the above notation consider $f_1(x_{i,\alpha}) \in \mathcal{H}_{\gamma_1}$ and $f_2(x_{i,\alpha}) \in \mathcal{H}_{\gamma_2}$. The product is given by the formula

$$f_1 \cdot f_2 = f_1\left((x'_{i,\alpha})\right) f_2\left((x''_{i,\alpha})\right) \sum \frac{\prod_{i,j\in I} \prod_{\alpha_1=1}^{\gamma_i^1} \prod_{\alpha_2=1}^{\gamma_j^2} \left(x''_{j,\alpha_2} - x'_{i,\alpha_1}\right)^{\mathrm{a_{ij}}}}{\prod_{i\in I} \prod_{\alpha_1=1}^{\gamma_i^1} \prod_{\alpha_2=1}^{\gamma_i^2} \left(x''_{i,\alpha_2} - x'_{i,\alpha_1}\right)} \in \mathcal{H}_\gamma \tag{1}$$

where the sum is over all possible shuffles.

Remark 3.2. If you are already familiar with torus localisation then the following comment may help parse Theorem 3.1. The proof is an application of the Bott localisation theorem. The sum is over torus fixed points; the denominator is a normal bundle contribution and the numerator an integral against a fixed point locus. See [KS11, Theorem 2].

4. Examples

4.1. **Quiver with one vertex.** Let Q be the quiver with one vertex and d edges. The underlying abelian group of $\mathcal{H}$ is independent of the number of edges and we think of this group as a direct sum

$$\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ...$$

where $\mathcal{H}_i$ are symmetric polynomials in i formal variables. Our task will be understanding the product.

Let S be the set of tuples

$$S = \{i_1 < ... < i_n, j_1 < j_2 < ... j_m\} = \{1, ..., n+m\}.$$

The formula of Theorem 3.1 simplifies to

$$f_1 \cdot f_2 = \sum_S f_1(x_{i_1}, ..., x_{i_n}) f_2(x_{j_1}, ..., x_{j_m}) \left(\prod_{k=1}^{n} \prod_{\ell=1}^{m} (x'_{j_\ell} - x_{i_k})\right)^{d-1}.$$

We have already observed that the multiplication in $\mathcal{H}$ does not preserve the cohomological grading. If instead we declare a polynomial of degree k in n variables has bidegree $(n, 2k + (1-d)n^2)$ then $\mathcal{H}$ becomes a bigraded algebra. In this situation $\mathcal{H}$ is commutative for d odd and supercommutative for d even.

Example 4.1. (d=1) If Q has a singe vertex and a single edge then Theorem 3.1 simplifies further. Identifying

$$\mathcal{H}_1 = \mathbb{C}[X]$$

and denoting $X^i = \phi_{2i}$ the product structure is the symmetric algebra on the ϕ_{2i}.

Example 4.2. (d=0) For $d = 0$ there is an identification between $\mathcal{H}$ and an exterior algebra generated by elements of bidegree $(1, 2k+1)$ for $k \in \{0, 1, ...\}$. The generators of this exterior algebra are the X^i in $\mathcal{H}_1$.

Remark 4.3. (Symmetric quivers) We say a Quiver is *symmetric* if the matrix a_{ij} is symmetric. The trick we used to put a grading on the cohomological Hall algebra of a single vertex can be generalised to put a $\mathbb{Z}^I \times \mathbb{Z}$ grading on the cohomological Hall algebra of any symmetric Quiver.

Indeed write

$$\mathcal{H} = \oplus_{(\gamma,k)} \mathcal{H}_{\gamma,k} \text{ where } \mathcal{H}_{\gamma,k} = H^{k-\chi_Q(\gamma,\gamma)}(\mathsf{B}G_\gamma).$$

The algebra $\mathcal{H}$ is not quite supercommutative, but one can modify the product structure by a sign so that the result is supercommutative. See [KS11, Section 2.6].

4.2. **The A_2 quiver.** Consider the quiver with two vertices and one edge. The associated cohomological Hall algebra has two subalgebras $\mathcal{H}_R$ and $\mathcal{H}_L$ corresponding to representations supported on vertices 2 and 1 respectively.

The subalgebras $\mathcal{H}_L$ and $\mathcal{H}_R$ are isomorphic to the cohomological Hall algebra of the one vertex quiver with no edges - that is an exterior algebra. The multiplication map then induces an isomorphism $\mathcal{H}_R \otimes \mathcal{H}_L \to \mathcal{H}$.

References

[KS11] Maxim Kontsevich and Yan Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants. *Commun. Number Theory Phys.*, 5(2):231–352, 2011. 1, 5