Shuffle realisation of Quantum Groups

The goal of my talk is to introduce the techonology of Shuffle Algebras. These are certain algebra structures on the ring of symmetric functions and they provide a concrete way of dealing with algebras appearing in Geometric Representation theory. In a way they are algebraic manifestation of the correspondence diagrams we all really like. As we will see in next few talks, the CoHA for quivers without potential is shuffle algebra while the CoHA of pre-projective algebra injects into the Shuffle algebra. On the other hand, the Classical Hall algebra of elliptic curve is also known to be a Shuffle Algebra. Although the definition becomes more natural in the religm of CoHA, historically these algebras appeared in relation with the study of Quantum Groups. In this talk we will go through this historically while ending with few interesting conjectures.

Definition 0.0.1 (Kac-Moody Lie Algebras)**.** Let I be a finite set, $C = (C_{ij})$ be the symmetric cartan matrix, that is a matrix with entries $c_{i,i} = 2$ and $c_{i,j} = c_{j,i} \in \{0, -1, \cdots\}$. Then to it, we can associate Kac's Moody lie algebra $\mathfrak{g}$ which is complex lie algebra with generators $e_i, h_i, f_i \forall\, i \in I$ with relations

- $[h_i, h_j] = 0 \forall\, i, j \in I$
- $[e_i, f_j] = \delta_{i,j} h_i \forall\, i, j \in I$, $[h_i, e_j] = c_{i,j} e_j, [h_i, f_j] = -c_{i,j} f_j \forall\, i, j \in I.$
- $(\mathrm{ad} e_i)^{1-c_{i,j}} e_j = 0, (\mathrm{ad} f_i)^{1-c_{i,j}} f_j = 0\ \forall\, i, j \in I.$

Given any Lie algebra $\mathfrak{g}$, it's representation theory is captured by an asociative algebra, the Universal enveloping algebra $\mathbf{U}(\mathfrak{g})$. Given two representation V and W of $\mathfrak{g}$, the tensor product $V \otimes W$ is a representation, while the dual V^* is also a representation. One way to see this is by noticing that $\mathbf{U}(\mathfrak{g})$ has a well behaved coproduct given by $\Delta : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g}) \otimes \mathbf{U}(\mathfrak{g})$ and so representation $\rho_{V \otimes W}$ is just $(\rho_V \otimes \rho_W)\Delta$. While the dual representation is captured by existence of antipode $S : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g})$ given by $S(x) = -x$ and then ρ_{V^*} is just $f \to S(x)f$. More generally, having structure of monoidal category on the category of representations requires a structure of Hopf algebra. Notice however that for representations of $\mathbf{U}(\mathfrak{g})$, we have an isomorphism $V \otimes W \to W \otimes V$ given by flip. This is captured by the fact that $\mathbf{U}(\mathfrak{g})$ is a cocommutative Hopf Algebra. The Drinfeld-Jimbo Quantum Groups are deformation of $\mathbf{U}(\mathfrak{g})$ as a Hopf algebra in a way that they are neither commutative, neither cocommutative but still not too far from being co-commutative.

We recall q analogs. $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]! = [n][n-1] \cdots [1]$ and finally $\binom{[n]}{[k]} = \frac{[n]!}{[k]![n-k]!}$.

Definition 0.0.2 (Drinfeld Jimbo Quantum Group [Sch09])**.** $\mathbf{U}_q(\mathfrak{g})$ is a $\mathbb{C}(q)$ algebra generated by

$$e_i, f_i, \phi_i^{\pm 1} \forall\, i \in I$$

satisfying the following relations:

$$\phi_j e_i = q^{c_{i,j}} e_i \phi_j$$
$$\phi_j \phi_k = \phi_k \phi_j$$
$$\sum_{k=0}^{1-c_{i,j}} (-1)^k \binom{[1-c_{i,j}]}{[k]} e_i^k e_j e_i^{1-c_{i,j}-k} = 0$$

as well as the opposite relations with e_i replaced with f_j namely

$$f_i \phi_j = q^{c_{i,j}} \phi_j f_i$$
$$\sum_{k=0}^{1-c_{i,j}} (-1)^k \binom{[1-c_{i,j}]}{[k]} f_i^k f_j f_i^{1-c_{i,j}-k} = 0$$

and the commutation rule

$$[e_i, f_j] = \delta_{ij} \cdot \frac{\phi_i - \phi_i^{-1}}{q - q^{-1}}$$

Remark 0.0.3. To see this as as a deformation of $\mathbf{U}(\mathfrak{g})$ one substitute $\phi_i = q^{h_i}$ and consider the above as an algebra over $\mathbb{C}[[h]]$ and take limit $q \to 1/$ [CP94]

Remark 0.0.4. We don't really need cartan matrices to be symmetric, it's possible to make sense of the definition for any symmetrizabe cartan matrix. All we need to do is to consider $q_i = q^{d_i/2}$ binomials.

This algebra is neither commutative nor co-commutative however its representation category satisfies following interesting propertly. The flip morphsim $\tau : V \otimes W \to W \otimes V$ isn't an isomorphism any more, i.e the representation category is not symmetric monoidal anymore, however it is braided. We have isomorphisms $R_{V,W} : V \otimes W \to W \otimes V$ such that they satisfy Yang-Baxter Equation

$$R_{U,V} R_{U,W} R_{V,W} = R_{V,W} R_{U,W} R_{U,V}$$

This is because $\mathbf{U}(\mathfrak{g})$ is an example of quasi-triangular Hopf algebra, beside being an hopf algebra, there exist an element $R \in H \hat{\otimes} H$ called the universal R matrix which satisfy

- $\forall\, x \in H,\, R\Delta(x) = \Delta^{\mathrm{op}}(x) R$
- $(\Delta \otimes \mathrm{id})(R) = R_{1,3} R_{2,3}$
- $(\mathrm{id} \otimes \Delta)(R) = R_{1,3} R_{1,2}$

It's easy to see that once we have element satisfying this properties, it satisfies the Yang-Baxter Equation

$$R_{1,2} R_{1,3} R_{2,3} = R_{2,3} R_{1,3} R_{1,2}$$

The quantum groups were first studied as source of solutions to this equation. It's a highly non-trivial theorem of Drinfeld that the Drinfeld-Jimbo quantum groups are quasi-triangular. The general technique to show the existence and compute R matrix is by Drinfeld double. The quantum groups $\mathbf{U}_q(\mathfrak{g})$ turn out to be double of $\mathbf{U}_q(\mathfrak{b})$.

Definition 0.0.5 (Drinfeld Double [Neg22a]). Given two Hopf algebras A^- and A^+ over base field $\mathbb{F}$, a bilinear form

$$A^- \otimes A^+ \to \mathbb{F}$$

It's a Hopf pairing if it satisfied

$$\langle aa', b\rangle = \langle a \otimes a', \Delta(b)\rangle \tag{1}$$

$$\langle a, bb'\rangle = \langle \Delta^{\mathrm{op}}(a), b \otimes b'\rangle \tag{2}$$

$$\langle Sa, b\rangle = \langle a, S^{-1}b\rangle \tag{3}$$

Given this pairing, the Drinfeld double is an Hopf algebra structure on

$$A = A^+ \otimes A^-$$

where the components interact using the rule

$$\langle a_1, b_1\rangle a_2 \cdot b_2 = b_1 \cdot a_1 \langle a_2, b_2\rangle$$

for all $a \in A^-, b \in A^+$ where $\Delta(a) = \sum a_1 \otimes a_2$

When the pairing is non-degenerate, it gives rise to canonical universal R matrix $\in A^- \hat{\otimes} A^+$, given by

$$R = \sum_i a_i \otimes (a_i)^*$$

where a_i is the basis.

So now we will only focus on the positive half $\mathbf{U}_q(\mathfrak{n}^+)$ of these algebras. To construct it's PBW basis more combinatorially, the work of Rosso-Green provided a new way to think about these algebras.

Definition 0.0.6 (Quantum Shuffle Algebra [NT21]). $\mathcal{F}_{\mathfrak{g}}$ is a algebra over $\mathbb{C}(q)$ with basis given by words

$$[i_1 \cdots i_k]$$

for arbitary $k \in \mathbb{N}$ and $i_k \in I$ with multplication rule given by the shuffle product

$$[i_1, \cdots i_k] * [j_1, \cdots j_l] = \sum_{\{1,\cdots,k+l\} = A \cup B, |A|=k, |B|=l} q^{\lambda_{A,B}} \cdot [s_1 \cdots s_{k+l}]$$

where if $A = \{a_1 < \cdots a_k\}$ and $B = \{b_1, \cdots b_l\}$, then we have $s_c = i_*$ is $c = a_* \in A$, otherwise $s_c = j_*$ where $c = b_* \in B$ and $\lambda_{A,B} = \sum_{A \in a > b \in B} c_{s_a, s_b}$.

These coefficients are set up in a way to have the following theorem

Theorem 0.0.7 (Rosso). There is an injective algebra morphism

$$\Phi : \mathbf{U}_q(\mathfrak{n}^+) \to \mathcal{F}_{\mathfrak{g}}$$

given by $\Phi(e_i) = [i]$.

To see what's going on, lets do some computations.

Example 1. $\mathfrak{g} = \mathfrak{sl}_3$ Then $\mathbf{U}_q(\mathfrak{n}^+)$ is a $\mathbb{C}(q)$ algebra generated by e_1, e_2 such that it satisfy the quantum Serre relation $e_2e_1^2 - (q+q^{-1})e_1e_2e_1 + e_1^2e_2 = 0$. The above map is well defined because

$$e_2e_1^2 \mapsto (1+q^2)([211] + q^{-1}[121] + q^{-2}[112])$$

$$e_1^2e_2 \mapsto [112] + q^2[112] + q[121] + q^{-1}[121] + q^{-2}[211] + [211]$$

$$e_1e_2e_1 \mapsto [121] + q^2[112] + q[121] + q^{-1}[112] + q[112] + [121]$$

Remark 0.0.8. This morphism is not a surjection however the image has a nice combinatorial description in terms of combinatorics of lydon-words and that gives a combinatorial PBW basis for the positive half of the quantum group.

Just like drinfeld jimbo quantum groups give rise to solution of Yang-Baxter equations, the affine quantum groups $\mathbf{U}_q(\widetilde{\mathfrak{g}})$ give rise to solution of parametrized Yang-Baxter. Notice that affine lie algebras $\widetilde{\mathfrak{g}}$ arises as Kac-Moody lie algebras, however one can also define them to be the central extension of Loop lie algebra $\mathcal{L}\mathfrak{g}$ which is a lie algebra with vector space $\mathfrak{g}[t, t^{-1}]$ and the lie bracket $[t^n a, t^m b] = t^{n+m}[a, b]$. Drinfeld and Beck made sense of this analogy to the level of quantum groups, giving rise to 'new realisation' of affine quantum groups. From now on we will focus on the positive half $\mathbf{U}_q^>(\mathcal{L}\mathfrak{g})$ for semisimple lie algebra $\mathfrak{g}$.

In quantum loop group, we have generators $e_{i,k} \forall\, i \in I, k \in \mathbb{Z}$, corresponding to $e_i t^k$. To write the relations in a vertex form, we package them into the generating series

$$e_i(z) = \sum_{k \in \mathbb{Z}} \frac{e_{ik}}{z^k}$$

Definition 0.0.9 (Quantum Loop Groups)**.** $\mathbf{U}_q^>(\mathcal{L}\mathfrak{g})$ is a $\mathbb{C}(q)$ algebra generated by $e_{i,k} \forall\, i \in I, k \in \mathbb{Z}$ such that we have

Vertex Relation for all $i, j \in I$

$$e_i(z)e_j(w)\zeta_{ji}\left(\frac{w}{z}\right) = e_j(z)e_i(w)\zeta_{ij}\left(\frac{z}{w}\right)$$

where

$$\zeta_{i,j}\left(\frac{z}{w}\right) = \frac{z - wq^{-c_{ij}}}{z - w}$$

Drinfeld-Serre Relation for all $i \neq j \in I$

$$\sum_{\sigma \in S_{1-c_{ij}}} \sum_{k=0}^{1-c_{i,j}} (-1)^k \binom{[1-c_{i,j}]}{[k]} e_i(z_{\sigma(1)}) \cdots e_i(z_{\sigma(k)}) e_j(w) e_i(z_{\sigma(k+1)}) \cdots e_i(z_{\sigma(1-c_{i,j})}) = 0$$

Remark 0.0.10. One can define Quantum Loop group for arbitary symmetrizable Kac's moody lie algebra but then imposing these relations aren't enough! [Neg23]. The resulting algebra is a deformation of $\mathcal{L}'\mathfrak{g}$ where the relations in the lie algebra $\mathcal{L}'\mathfrak{g}$ are meant to immitate $\mathcal{L}\mathfrak{g}$ however in general there is only a surjection $\mathcal{L}'\mathfrak{g} \to \mathcal{L}\mathfrak{g}$ with kernel trivial iff $\mathfrak{g}$ is semisimple [Sch09][Prop A.19].

Just like before one would like to understand the Positive Half more combinatorially. By considering a degeneration of Feigin-Odesski Shuffle algebra, In [Enr98] Enrique's gave an explicit algebra map from the Quantum group to a Shuffle algebra, an algebra defined on the space of Symmetric Laurent polynomials.

Example 2. Let us explain the construction for the case of $\mathfrak{sl}_2$. We consider algebra on the vector space $\mathcal{V} = \sum_{k \in \mathbb{N}_{\geq 0}} \mathbb{Q}(q)[z_1^{\pm}, \cdots, z_k^{\pm})^{S_k}$ such that shuffle multiplication is defined by

$$F(z_1, \cdots, z_k) * G(z_1, \cdots, z_l) = \frac{\mathrm{Sym}(F(z_1, \cdots, z_k)G(z_{k+1} \cdots, z_{k+l})}{k! \cdot l!} \prod_{1 \leq a \leq k, k+1 \leq b \leq l} \zeta_{1,1}\left(\frac{z_a}{z_b}\right)$$

We can then define $\Psi : \mathbf{U}_q^>(\mathcal{L}\mathfrak{g}) \to \mathcal{V}$ defined by $e_{i,k} \mapsto [z_1^k]$ which is same as sending

$$e(z) \to \sum_{n \in \mathbb{Z}} \frac{z_1^n}{z^n} = \delta\left(\frac{z_1}{z}\right)$$

So to check that this map is well defined, all we need to make sure is that

$$\delta\left(\frac{z_1}{z}\right) * \delta\left(\frac{z_1}{w}\right)(q^2w - z) = -\delta\left(\frac{z_1}{w}\right) * \delta\left(\frac{z_1}{z}\right)(q^2z - w)$$

This follows since LHS is

$$\sum_{n,m} z^{-n}w^{-m}\,\mathrm{Sym}\left(z_1^n z_2^{m+1}\frac{(q^2+q^{-2})z_1^{n+1}z_2^{m+1} - (z_1^n z_2^{m+2} + z_1^{n+2}z_2^m)}{z_1 - z_2}\right)$$

which is anti-symmetric.

In general this morphism is proven to be injection by Enriques and the image of morphism are the polynomials satisfying the wheel condition.

This is massively generalized in [Neg22b]. We now give a general definition of Shuffle algebra for arbitary Quiver. The above algebra is one specific specialization of the Shuffle algebra.

Consider any arbitary quiver Q with vertices I and edges E. We consider the setting corresponding to largest possibe torus action. Consider $\mathbb{F} = \mathbb{Q}(q, t_e)_{e\in E}$ We consider the vector space

$$\mathcal{V}_Q = \bigoplus_{\mathbf{d}=(d_i)_{i\in I}\in\mathbb{N}^I} \mathbb{F}[\dots, z_{i1}^{\pm}, \cdots, z_{in_i}^{\pm}, \cdots]^{\mathrm{Sym}}$$

where we are considering laurent polynomials which are symmetric in color i, that is for each $i \in I$, the variables $z_{i,1}, \cdots, z_{i,n_i}$ are symmetric and we define the shuffle product in similar way as before

$$F(\dots, z_{i1}, \dots, z_{in_i}, \dots) * F'(\dots, z_{i1}, \dots, z_{in'_i}, \dots) =$$
$$\mathrm{Sym}\left[\frac{F(\dots, z_{i1}, \dots, z_{in_i}, \dots)F'(\dots, z_{i,n_i+1}, \dots, z_{i,n_i+n'_i}, \dots)}{n!n'!}\prod_{1\le a\le n_i, n_j<b\le n_j+n'_j}^{i,j\in I}\zeta_{ij}\left(\frac{z_{ia}}{z_{jb}}\right)\right]$$

where for $i, j \in I$,

$$\zeta_{ij}(x) = \left(\frac{1-xq^{-1}}{1-x}\right)^{\delta_{ij}}\prod_{e=ij\in E}\left(\frac{1}{t_e} - x\right)\prod_{e=ji\in E}\left(1 - \frac{t_e}{qx}\right)$$

we symmetrize each variable seperately.

We would like to consider algebra generated by the degree 1 elements. This natural give rise to a subalgebra satisfying wheel conditions.

Definition 0.0.11 (Shuffle Algebra)**.** The shuffle algebra $\mathcal{S}_Q \subset \mathcal{V}_Q$ is defined to be the subset of laurent polynomials $F(\dots, z_{i1}, \dots, z_{in_i}, \dots)$ which satisfy the wheel conditions

$$F|_{z_{ia}=\frac{qz_{jb}}{t_e}=qz_{ic}} = F|_{z_{ja}=t_ez_{ib}=qz_{jc}} = 0$$

for all edges $e =\!\rightarrow ij$ and all $a \neq c$ and $a \neq b \neq c$ if $i = j$.

Remark 0.0.12. From the definition, one can easily see that wheel conditions forms a subalgebra and so the subalgebra generated by $z_{i,j}^d$ is contained inside $\mathcal{S}_Q$. However the converse is also true. That is shuffle algebra is spherically generated. This implies that Enriqures morphism is a surjection. The proof is complicated and involves Negut's combinatorics of words[NT21].

Remark 0.0.13. It's possible to add cartan elements to the shuffle algebra making it a Hopf algebra and there is a natural pairing allowing us to consider its double, using the drinfeld double technology we explained before. Let $\mathcal{A}_Q$ be the resulting algebra. It is defined so that $\mathcal{A}_Q^{\geq} = \mathcal{S}_Q$ while $\mathcal{A}_Q^{-} = \mathcal{S}_Q^{\mathrm{op}}$.

Remark 0.0.14 (Relation with K theoritic Hall algebras)**.**

- The Cohomological Hall algebra structure on the cohomology of moduli of quiver representations can be lifted to K theoritic setup. Lets call resulting algebra K_Q. Then it can be shown that there is an algebra isomorphism

$$K_{Q^{\mathrm{double}}} \simeq \mathcal{V}_Q^{\mathrm{int}}$$

where $\mathcal{V}_Q^{\mathrm{int}}$ is above algebra considered with coefficient in laurent series, while the $K_{Q^{\mathrm{double}}}$ is considered with the torus action $T = \mathbb{C}^* \times \prod_{e \in E} \mathbb{C}^*$ acting by scaling the arrow e by t_e and e^* by q/t_e.

- We will learn a similar algebra structure on K theory of moduli of representations of pre-projective algebra. Lets call the resulting algebra K_{Π_Q}. The moduli of representations of pre-projective algebra can be thought of as moduli of represenatations of double quiver with relations. This gives an map from K_{Π_Q} to $K_{Q^{\mathrm{double}}}$. This map is known to be injection while when localized it's image is exactly the laurent polynomials satisfying wheel condition. That is

$$K_{\Pi_Q}^{\mathrm{loc}} \simeq \mathcal{V}_Q$$

Remark 0.0.15 (Relation with classical Hall algebra of curves)**.** The shuffle algebra for the Jordan Quiver turn out to be positive half of the Elliptic Hall algebra($t_e \to q, q/t_e \to \overline{q}$) [Neg14]. In general the shuffle algebra of g- loop quiver can be specialized to give spherical part of Classical hall algebra of curve of genus g. [NSS21].

The shuffle algebra product is consistent with the grading given by the dimension vector and homogenous degree. This gives $\mathbb{N}^I \times \mathbb{Z}$ grading on the algebra via $\deg(F) = (\mathbf{n}, d)$. We think of $\mathbf{n}$ as the horizontal grading while d being the vertical component. This allows us to give a notion of slope.

We have two pairings $\mathbf{k} \cdot \mathbf{l} = \sum k_i l_i$ while $\langle \mathbf{k}, \mathbf{l} \rangle = \sum_{i,j} k_i l_j n_{ij}$ where n_{ij} are the number of arrows ij.

Definition 0.0.16 (Naive slope)**.** We say that $F \in \mathcal{S}_{\mathbf{n},d} \subset \mathcal{S}_Q$ has naive slope $\leq m \in \mathbb{Q}^I$ if $d \leq \mathbf{m} \cdot \mathbf{n}$.

Its possible to refine the notion of slope

Definition 0.0.17 (Slope)**.** Let $\mathbf{m} \in \mathbb{Q}^I$. Then F has slope $\leq \mathbf{m}$ if

$$\lim_{\zeta \to \infty} \frac{F(\ldots, \zeta z_{i1}, \ldots, \zeta z_{ik_i}, z_{i,k_i+1}, \ldots)}{\zeta^{\mathbf{m} \cdot \mathbf{k} + \langle \mathbf{k}, \mathbf{n} \text{-} \mathbf{k} \rangle}}$$

is finite $\forall \, \mathbf{0} \leq \mathbf{k} \leq \mathbf{n}$. similarly one can define slope for $G \in \mathcal{A}_Q^{-}$.

The slope elements share the following interesting propertly. The coproduct doesn't respect the slope however one has that element $F \in \mathcal{A}^+$ if and only if $\Delta(F) = (\text{anything}) \otimes (\text{naive slope} \leq m)$. This property also shows that subspace of elements of slope $\leq \mathbf{m}$ forms a subalgebra.

Definition 0.0.18 (Slope algebra)**.** For any $\mathbf{m} \in \mathbb{Q}^I$ we define

$$\mathcal{B}_{\mathbf{m}}^{\pm} \subset \mathcal{A}^{\pm}$$

for the graded subalgebra of elements of slope $\leq \mathbf{m}$ and naive slope $= \mathbf{m}$. One can then again extend to $\mathcal{B}_{\mathbf{m}}^{\geq}$ and $\mathcal{B}_{\mathbf{m}}^{\leq}$ forming a Hopf algebra with the coproduct $\Delta_{\mathbf{m}}$ defined by considering the leading naive slope terms of the coproduct Δ. Then we can restrict the bilinear form from the Shuffle algebra and take the Drinfeld double to finally have the slope algebra $\mathcal{B}_{\mathbf{m}}$. Then $\mathcal{B}_{\mathbf{m}}$ forms a subalgebra of $\mathcal{A}_Q$.

Remark 0.0.19. The algebras $\mathcal{B}^{\pm}_{\mathbf{m}}$ are subalgebras of $\mathcal{A}^{\pm}$ by the definition the coproduct on $\mathcal{B}^{\geq}$ is not the same as the co-product on $\mathcal{A}^{\geq}$. This means that apriori the drinfeld double $\mathcal{B}_{\mathbf{m}}$ is not a subalgebra of $\mathcal{A}_Q$.

Example 3 (Slope algebras for Cyclic quivers [Neg15])**.** For cylic quiver of length n,

$$\mathcal{B}_{(m_1,\dots,m_n)} = \mathbf{U}_q(\hat{\mathfrak{gl}}_{n_1}) \otimes \cdots \otimes \mathbf{U}_q(\hat{\mathfrak{gl}}_{n_d})$$

where n_i are defined by a combinatorial procedure and $\mathcal{B}_{\mathbf{0}} = \mathbf{U}_q(\hat{\mathfrak{gl}}_n)$.

In the way analogus to factorization of elliptic hall algebra into slopes coming by slope of coherent sheaves, The slope sub algebras allows to break shuffle algebra into smaller as we have

Theorem 0.0.20 (Factorization of Shuffle algebra [Neg22a])**.** For any $\mathbf{m} \in \mathbb{Q}^I$ and $\theta \in \mathbb{Q}^I_+$, we have isomorphism given by multiplication

$$\overset{\rightarrow}{\bigotimes_{r\in\mathbb{Q}}} \mathcal{B}^+_{\mathbf{m}+r\theta} \simeq \mathcal{S}_Q$$

This factorization passes to the R matrices and gives a factorization of R matrix.

Let's consider the size of $\mathcal{B}_{\mathbf{0}}$ by recording the graded dimension. We define

$$\chi_{\mathcal{B}_{\mathbf{0}}}(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{N}^I} \dim \mathcal{B}_{\mathbf{0}|\mathbf{n}} \mathbf{z}^{\mathbf{n}}$$

Example 4 (Jordan Quiver)**.** In this case it turns out that $\mathcal{B}_0 \simeq \mathbf{U}_q(\hat{\mathfrak{gl}}_1)$ which implies that

$$\chi_{\mathcal{B}_0}(z) = \sum_n p(n) z^n = \prod_d \frac{1}{1-z^d} = \mathrm{Exp}(A_{Q_{\mathrm{Jor}}}(1,z))$$

Then we have the following conjecture. For any quiver Q, we have:

$$\chi_{\mathcal{B}_{\mathbf{0}}}(\mathbf{z}) = \mathrm{Exp}(A_Q(1,\mathbf{z}))$$

where $A_Q(t,\mathbf{z}) = \sum_{\mathbf{d}\in\mathbb{N}^I\backslash\mathbf{0}} A_{Q,\mathbf{d}}(t)\mathbf{z}^{\mathbf{d}}$ and $A_{Q,\mathbf{d}}(t)$ is the Kac's polynomial counting the number of isomoprhism classes of $\mathbf{d}$ dimensional absoutely indeceomposable represenation of the quiver Q over a finite field with t elements and

$$\mathrm{Exp}\left[\sum_{\mathbf{n}\in\mathbb{N}^I\backslash\mathbf{0}} d_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}\right] := \prod_{\mathbf{n}\in\mathbb{N}^I-0} \frac{1}{(1-\mathbf{z}^{\mathbf{n}})^{d_{\mathbf{n}}}}$$

Remark 0.0.21. A possible way to approach or to think about this conjecture is the following. In the later talks we will see that the Cohomological Hall algebra for the pre-projective algebra has a lie algebra sitting inside it. Also it's known that Poincare polnomial of BPS lie algebra is the Kac's polynomial. Thus the conjectue will follow if we show that the degeneration map from K theory to cohomology sends $\mathcal{B}_{\mathbf{0}}$ to $\mathbf{U}(\mathfrak{g}_{\mathrm{BPS}})$.

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