VERTEX ALGEBRAS, DEFINITION AND MOTIVATIONS

1. Formal power series and notation

Let V be a $\mathbb{K}$-algebra. We define

$$V[[z]] = \left\{ \sum_{n\in\mathbb{N}} a_n z^n \mid a_n \in V \right\},$$

$$V[[z^{\pm}]] = \left\{ \sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in V \right\},$$

$$V((z)) = \left\{ \sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in V, a_n = 0 \text{ for sufficiently small } n \right\},$$

and analogously for several variables. Elements of $V[[z]]$, $V[[z^{\pm}]]$ and $V((z))$ are denoted $a(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, and we sometimes want to write them as $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, i.e. $a_{(n)} = a_{-n-1}$. One can define the derivative of a formal power series as

$$\partial_z \left(\sum_{n\in\mathbb{Z}} a_n z^n \right) := \sum_{n\in\mathbb{Z}} n a_n z^{n-1}.$$

While there is no well-defined multiplication of formal power series, one can define a product of two series in different formal variables as the map $V[[z^{\pm}]] \times V[[w^{\pm}]] \to V[[z^{\pm}, w^{\pm}]]$ given by

$$\left(\sum_{n\in\mathbb{Z}} a_n z^n, \sum_{n\in\mathbb{Z}} b_n w^n \right) \mapsto \sum_{n,m\in\mathbb{Z}} a_n b_m z^n w^m$$

One can also always define a product of a formal power series by a Laurent polynomial as the map $V[z^{\pm}] \times V[[z^{\pm}]] \to V[[z^{\pm}]]$ given by

$$\left(\sum_{n=n_{min}}^{n_{max}} a_n z^n, \sum_{m\in\mathbb{Z}} b_m z^n \right) \mapsto \sum_{k\in\mathbb{Z},} c_k z^k, \text{ where } c_k = \sum_{n=n_{min}}^{n_{max}} a_n b_{k-n}.$$

In particular, for any $a(z) \in V[[z^{\pm}]], b(w) \in V[[w^{\pm}]], N \in \mathbb{N}$ the expression

$$(z-w)^N [a(z), b(w)] := (z-w)^N (a(z)b(w) - b(w)a(z))$$

makes sense.

Remark 1.1. In most references $\mathbb{K}$ is assumed to be the field of complex numbers $\mathbb{C}$ [AM23, FBZ04, Kel17], but [FHL93, Joy21] assumes $\mathbb{K}$ is any field of characteristic 0. V is usually assumed to be a $\mathbb{K}$-vector space [FHL93, Kel17, Joy21], or a $\mathbb{K}$-algebra [AM23, FBZ04]. When V is a vector space it clearly doesn't make sense to "multiply" power series, or even polynomials with coefficients in V, as there is no multiplication on V.

2. Vertex algebras

Let $\operatorname{End} V$ be the $\mathbb{K}$-algebra of linear endomorphisms of V. For any $b \in V$, a power series $a(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in \operatorname{End} V[[z^{\pm}]]$ defines a power series $a(z)b \in V[[z^{\pm}]]$, given by $\sum_{n\in\mathbb{Z}}(a_n b)z_n$, where $a_n b$ is the value of the endomorphism a_n on the vector b.

Remark 2.1. From now on $a, b, \ldots$ will denote vectors in V and $a(z), b(z), \ldots$ the corresponding power series in $\operatorname{End} V[[z^{\pm}]]$. In particular $a_n, b_n, \ldots$ are elements of $\operatorname{End} V$, and so are $a_{(n)} = a_{-n-1}$.

A *field on* V is an element $a(z) \in \operatorname{End} V[[z^{\pm}]]$ such that for any $b \in V$ the power series $a(z)b$ lies in $V((z))$, i.e. it has only finitely many non-zero coefficients at negative powers of z; for any $b \in V$ there exists an $N \in \mathbb{N}$ such that $a_{-n}b = 0$ for all $n \geq N$. The set $\mathcal{F}(V)$ of all fields on V is naturally a $\mathbb{K}$-vector space. Two fields $a(z), b(z) \in \mathcal{F}(V)$ are called *local with respect to each other* if there exists an $N \in \mathbb{N}$ such that $(z-w)^N[a(z), b(w)] = 0$.

Given two fields $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, b(w) = \sum_{n\in\mathbb{Z}} b_{(n)} w^{-n-1} \in \mathcal{F}(V)$, their *normal ordered product* $: a(z)b(w) :$ is defined as

$$\sum_{n\in\mathbb{Z}} \left(\sum_{m<0} a_{(m)} b_{(n)} z^{-m-1} + \sum_{m\geq 0} b_{(n)} a_{(m)} z^{-m-1} \right) w^{-n-1}.$$

Definition 2.2. A *vertex algebra* is a $\mathbb{K}$-vector space V together with

- a distinguished vector $|0\rangle \in V$, called the *vacuum vector*,
- a linear map $T : V \to V$, called the *translation operator*,
- a linear map $Y : V \to \mathcal{F}(V)$, called the *vertex operator* or *state-field correspondence*, taking $a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$.

satisfying the following axioms

VA1 (*vacuum axiom*) $Y(|0\rangle, z) = \mathrm{id}_V$, and for any $a \in V$, $Y(a, z)|0\rangle \in a + zV[[z]]$.

VA2 (*translation axiom*) $T|0\rangle = 0$ and for any $a \in V$

$$[T, Y(a, z)] = \partial_z Y(a, z).$$

VA3 (*locality axiom*) For any $a, b \in V$, there exists an $N \in \mathbb{N}$ such, that

$$(z-w)^N[Y(a, z), Y(b, w)] = 0,$$

i.e., fields $Y(a, z), Y(b, z)$ are local with respect to each other.

A *morphism of vertex algebras* is a linear map $\phi : V \to W$ such that $\phi(|0\rangle_V) = |0\rangle_W$, $\phi \circ T_V = T_W \circ \phi$ and $\phi \circ Y_V = Y_W \circ \phi$. Vertex algebras together with morphisms of vertex algebras form a category.

Given two vertex algebras V, W one can equip the tensor product $V \otimes W$ with a vertex algebra structure.

Remark 2.3. In VA2, multiplication of T and $Y(a, z)$ means pre- or post-composing the coefficients $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ with T, so that

$$[T, Y(a, z)] := \sum_{n\in\mathbb{Z}} (T \circ a_{(n)}) z^{-n-1} - \sum_{n\in\mathbb{Z}} (a_{(n)} \circ T) z^{-n-1}.$$

Computing $[T, Y(a, z)]$ at $|0\rangle$ and using the translation axiom gives $Ta = a_{(-2)}|0\rangle$, so one can get rid of T and the translation axiom in the definition of a vertex algebra.

Remark 2.4. There is a graded version of the definition of a vertex algebra, in which V is assumed to be a graded vector space and the structure of a vertex algebra on V in "compatible with the grading". Most common gradings are over $\mathbb{Z}$ ([BLM22, Joy21, FHL93]), $\mathbb{Z}_+$ ([FBZ04]) and $\mathbb{Z}_2$ ([Kac98]). Sometimes an additional assumption that all graded parts of V are finite-dimensional ([FHL93]) is added.

3. Virasoro Lie algebra

The Virasoro Lie algebra is the Lie algebra $\mathcal{L}$ with basis $\{C\} \cup \{L_n\}_{n\in\mathbb{Z}}$ and Lie bracket

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{m(m^2-1)}{12}\delta_{n+m,0}C,$$

where $\delta_{i,j}$ is equal to 1 if $i=j$ and 0 otherwise. We will use it to construct an example of a vertex algebra, and as motivation for the definition of a vertex operator algebra.

First, decompose $\mathcal{L}$ into a direct sum of graded parts

$$\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_{(n)}, \text{ where } \mathcal{L}_{(0)} = \mathbb{K}L_0 \oplus \mathbb{K}C \text{ and } \mathcal{L}_{(n)} := \mathbb{K}L_{-n} \text{ for } n \neq 0,$$

and define $\mathcal{L}_{(\leq 1)} := \bigoplus_{n\leq 1} \mathcal{L}_{(n)}$.

Given $c \in \mathbb{K}$, one can construct a represtentation Vir^c of $\mathcal{L}$, as follows. Let $\mathbb{K}|0\rangle_c$ be a one-dimensional $\mathcal{L}_{(\leq 1)}$-module, with the module structure given by

$$C|0\rangle_c = c|0\rangle_c, \quad L_n|0\rangle_c = 0.$$

Let $\mathcal{U}(\mathcal{L})$ be the universal enveloping algebra of $\mathcal{L}$, and $\mathcal{U}(\mathcal{L}_{(\leq 1)})$ the universal enveloping algebra of $\mathcal{U}(\mathcal{L}_{(\leq 1)})$. Define

$$Vir^c := \mathcal{U}(\mathcal{L}) \otimes_{\mathcal{U}(\mathcal{L}_{(\leq 1)})} \mathbb{K}|0\rangle_c.$$

As a vector space, Vir^c has a basis consisting of elements of the form

$$L_{-n_1} \dots L_{-n_k}|0\rangle_c, \text{ where } n_1 \geq \dots \geq n_k \geq 2.$$

The $\mathcal{L}$-module structure is given by "multiplication on the left", together with the conditions describing the action of $\mathcal{L}_{(\leq 1)}$ on $\mathbb{K}|0\rangle_c$ and the commutation relations in $\mathcal{L}$.

4. Vertex operator algebras

Vertex operator algebras, also called conformal vertex algebras, consist of a vector space V together with some structure built on top of the vertex algebra structure - in particular a vertex operator algebra is also a vertex algebra.

Definition 4.1. A vertex operator algebra is a vertex algebra $(V, |0\rangle, T, Y)$ together with

- a $\mathbb{Z}_+$-grading on $V = \bigoplus_{n\in\mathbb{Z}} V_{(n)}$, with $\dim V_{(n)} < \infty$,
- a distinguished vector $\omega \in V$, called the *conformal vector*

such that the coefficients of the field $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ satisfy the following axioms

VOA1 $[L_n, L_m] = (n-m)L_{n+m} + \frac{m(m^2-1)}{12}\delta_{n+m,0}c$, for some $c \in \mathbb{K}$.
VOA2 For $a \in V_{(n)}$, $L_0 a = na$.
VOA3 $L_{-1} = T$.

It follows from VA1 that $L_n|0\rangle = 0$ for $n \geq -1$, so VOA2 gives $|0\rangle \in V_{(0)}$.

Remark 4.2. Again, there are different approaches to gradation. Some references use gradation in $\mathbb{Z}_+$ as above ([FBZ04]), some use gradation in $\mathbb{Z}$ with the additional assumption that gradations below some n are 0 ([AM23, Kel17, FHL93, BLM22]). The conformal vector ω is sometimes assumed to be in gradation 2 ([FBZ04]), or 4 ([BLM22]). The gradation on V is sometimes assumed to be compatible with the vertex algebra structure, i.e. V is assumed to be a graded vertex algebra ([BLM22, AM23]).

Example 4.3. Let Vir^c be the representation of the Virasoro Lie algebra which we constructed in Section 3. Recall that Vir^c has a basis consisting of vectors of the form $L_{-n_1} \dots L_{-n_k}|0\rangle_c$, where $n_1 \geq \dots \geq n_k \geq 2$. It has a structure of a vertex operator algebra given by:

- $|0\rangle := |0\rangle_c$,
- $T := L_{-1}$
- $Y(L_{-n_1} \dots L_{-n_k}|0\rangle_c, z) := \frac{1}{(n_1-2)! \dots (n_k-2)!} : \partial_z^{n_1-2} T(z) \dots \partial_z^{n_k-2} T(z) :$, where $: - :$ denotes the normal ordered product as in Section 2.
- $\omega := L_{-2}|0\rangle_c$,

and the grading on V is determined by the conditions $\deg|0\rangle = 0, \deg L_{-n}|0\rangle = -n$.

5. Lie algebra associated to a vertex algebra

There is a natural (=functorial) way to construct a Lie algebra from a vertex algebra. We follow the exposition (and the notation) in [FBZ04].

Let $(V, |0\rangle, T, Y)$ be a vertex algebra. Consider the vector space $V \otimes \mathbb{K}[t^{\pm}]$, and let

$$\partial : V \otimes \mathbb{K}[t^{\pm}] \to V \otimes \mathbb{K}[t^{\pm}], \quad \partial = T \otimes \mathrm{id} + \mathrm{id} \otimes \partial_t.$$

We will define a Lie algebra structure on the vector space $U'(V) := \operatorname{coker} \partial$. This vector space is spanned by elements of the form $a_{[n]} := q(a \otimes t^n)$, where q is the canonical map to the cokernel, subject to the relations $(Ta)_{[n]} = -na_{[n-1]}$. We define the Lie bracket on $U'(V)$ by the following formula

$$[a_{[m]}, b_{[k]}] := \sum_{n \geq 0} \binom{m}{n} (a_{(n)}b)_{[m+k-n]}.$$

Theorem 5.1. *The bracket defined above defines a Lie algebra structure on $U'(V)$, and the map $U'(V) \to \mathrm{End}(V)$ given by $a_{[n]} \mapsto a_{(n)}$ is a Lie algebra homomorphism.*

References

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