GLOBAL DT SHIFT AND LENS SPACES

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Motivation:

DT invariants first defined for moduli of coherent sheaves on CY3-folds

X - moduli space

Can virtual fundamental class $[X]_{vir}$

$$DT = \int_{[X_{vir}]} 1$$

Behrend

$$DT = \chi_{wt}(\nu_X) \qquad \nu_X \text{ - constructible function}$$

$$X = \mathrm{Crit}\, f \qquad f : \underset{\substack{\uparrow \\ \text{smooth}}}{U} \to \mathbb{C}$$

$$\nu_X(p) = (-1)^{\dim U} \chi(\underbrace{\phi_f \mathbb{Q}[\dim U]}_{\text{DT invariant}})(p)$$

Interested in

$\mathrm{Loc}_G(M)$ for M a 3-manifold

The derived versions of moduli spaces carry a -1 shifted symplectic structure

Durboux theorem $\Rightarrow$ can express the classica truncation locally as a critical locus

Thm X is a -1 shifted symplectic derived scheme, then $t_0(X)$ has the structure of a d-critical l

Def (X, s) X a scheme is a d-critical locus if for all points $p \in X$, there is a critica chart $C = (R, U, f, i)$

$p \in R$ - Zariski open

$f : U \to \mathbb{C}$
smooth closed embedding

$i : R \hookrightarrow U$ $\quad i(R) = \operatorname{crit} f$

$s \in S^0_X(X) \leftarrow$ won't define $\quad s|_R = f + I^2 \quad$ I ideal defin crit

" s remembers intrinsic information about f "

Example:

$U = \operatorname{Spec} \mathbb{C}[t^{\pm 1}, x] = \mathbb{C}^\times \times \mathbb{C}$

$f = x^n \quad g = t x^n \quad n \geq 2$

Crit f — Crit g

$$\operatorname{Spec} \frac{\mathbb{C}[t^{\pm 1}, x]}{(x^{n-1})} \qquad \operatorname{Spec} \frac{\mathbb{C}[t^{\pm}, x]}{(x^n, tx^{n-1})}$$

$n=2$

$\phi_f \mathbb{Q} = \mathbb{Q}[1]$ $\quad \phi_g \mathbb{Q} = L[1]$ $\quad L$ non-trivial local system

$n \geq 3$ $\quad s = f + I^2 = g + I^2$

$$\underline{x^n} + (x^{2n-2}) = \underline{tx^n} + (x^{2n-2})$$

<u>Example</u>

U smooth

$U \xrightarrow{0} \mathbb{C}$ $\qquad S^0_U = 0$

$U = \operatorname{Crit} 0$

$(U, 0)$

Canonical bundle and orientations

<u>Thm</u> (X, s) d-critical locus

Then $\exists$ a line bundle K_X on $X|_{red}$

s.t. for any chart $C = (R, U, f, i)$

we have $K_X|_R \cong i^* \omega_U^{\otimes 2}$

X - ' sh

$K_{t(x)} \cong$

An orientation is a line bundle $K_X^{1/2}$ and

a choice of isomorphism $(K_X^{1/2})^{\otimes 2} \cong K_X$

a choice of isomorphism $\varphi : (K_X) \to K_X$

Example:

U be smooth

$(U, 0)$

$$K_U \cong \omega_U^{\otimes 2}$$

There always exists an orientation since we can take $K_U^{1/2} = \omega_U$

U is the classical truncation of the derived crit locus and $\omega_U^{\otimes 2}$ comes from this.

Perverse sheaf

Let us denote by $PV_{U,f}$ the sheaf of vanishing cycles on $X = \mathrm{crit}\, f$

In general this is not the same as $\phi_f Q$.

Need to compare on overlaps

This is done by using embeddings of critical charts $\underset{R_1 U_1 f_1 i_1}{C_1} \xrightarrow{\varphi} \underset{R_2 U_2 f_2 i_2}{C_2}$

Can be shown that

$PV_{U_1,f_1} \cong \Phi \, PV_{U_2,f_2} \otimes P_\Phi$

P_Φ is a principal $\mathbb{Z}/2\mathbb{Z}$ bundle

means tensoring by associated rank 1 local system.

$\pi_1(R) \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^\times$ giving a local system

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(X,s) oriented $\varphi : (K_X^{1/2})^{\otimes 2} \to K_X$

Q_C orientation $\mathbb{Z}/2\mathbb{Z}$ bundle for each chart $C = (U, R, f, i)$

Local sections are local isomorphisms

$\alpha : K_X^{1/2}|_R \to i^* W_U$ s.t. $\alpha^{\otimes 2} = \varphi$

Thm (X,s) d-critical locus

then $\exists$ a perverse sheaf P s.t.

for any chart C

$$P|_R \cong i^* PV_{U,f} \otimes Q_C$$

Extension to stacks

-1 shifted symplectic derived Artin stack X $\rightsquigarrow$ d critical stack t_0

$\det \mathbb{L}_X \rightsquigarrow K_{t_0(X)} \cong \det(\mathbb{L}_X)|_{t_0(X)}$

Perverse sheaf

$\exists$ a perverse sheaf on X (X, s) d critical stack oriented

s.t. $P_X(T \to X)$ $\underset{\text{scheme}}{T} \xrightarrow{\text{smooth}} X$

$= P_T$

Stack of local systems on a 3-manifold.

G - reductive alg. gp.

Moduli of local systems on M
moduli of homomorphisms $\pi_1(M) \to G$ $\Big\}$ $Loc_G(M)$

Derived stack:

$RLoc_G(M) = Map(M_B, BG)$

M_B - Betti stack, which is the constant stack sending $\underset{\text{cdga}}{R}$ to singular simplicial set

PTVV: shifted symplectic structures for mapping stacks

X, Y are derived stacks

① Y is n-shifted symplectic

② X $\mathcal{O}$-compact

③ X has an orientation with parameter d

$Map(X,Y)$ has a $n-d$ shifted symplectic str

For manifolds M

$\mathcal{O}$-compactness $\leftrightarrow$ M is compact

Orientation d $\leftrightarrow$ M being orientable

$d = \dim M$

BG has only 2-shifted symplectic structures, which correspond to $Sym^2 \mathfrak{g}^{*G}$

$RLoc_G(M)$

$= Map(M_B, BG)$ is $2-\dim M$ shifted symplect

If $\dim M = 3$

Lens spaces $G = GL_n$

$L(m,n) = S^3/\mathbb{Z}/m\mathbb{Z}$ m,n coprime

$\parallel$

M Action generated by

$(z_1, z_2) \mapsto (z_1 \cdot e^{\frac{2\pi i}{m}}, z_2 e^{2\pi i n/m})$

Independent of n

$\pi_1(M) = \mathbb{Z}/m\mathbb{Z} = H$

$K = \mathbb{C}$

Group algebra of H is $KH = \frac{K[t]}{(t^m - 1)} = \frac{K[t]}{(t-1)} \times \cdots \frac{K[t}{(t-z^{m}}$

z mth root of unity

$KH = K^m$

To give a representation means giving

$$V = \bigoplus_{i=1}^{m} V_i$$

Fix a dimension x

$$\mathrm{Loc}_{GL_n}(M) = \coprod_{\substack{v_1 + \dots v_m \\ = x}} B\left(\prod_{i=1}^{m} GL_{v_i}\right)$$

D critical structure for quotient stacks

$$X = [U/G]$$

(X, s) $\longleftrightarrow$ G equivariant
d-critical locus structure on U

K_X $\leadsto$ G - equiv. line bundle L
and an iso
$$L \to K_U \otimes (\det \mathbb{L}_{U/X})^{-\otimes 2}$$

$K_X^{1/2}$ $\leadsto$ G - equiv line bundle L'
with an isomorphism
$$\varphi : L'^{\otimes 2} \to K_U \otimes (\det \mathbb{L}_{U/X})^{-\otimes 2}$$

For $BG = [pt/G]$

$\cdot\, U = pt \xrightarrow{0} \mathbb{C}$

$$K_U = \omega_U^{\otimes 2} \cong K$$

Orientation

$$\varphi: L^{\otimes 2} \to \omega_U \otimes (\det \mathbb{L}_{pt/BG})^{-\otimes 2}$$
$$= K \otimes (\det \mathfrak{g}^*)^{-\otimes 2}$$
$$= K \otimes (\det \mathfrak{g})^{\otimes 2}$$

For reductive groups $\det \mathfrak{g}$ is the trivial rep.

So $\varphi : L^{\otimes 2} \to K$

Fixing $L = K$

$\varphi \leftrightarrow$ a choice of $\sqrt{\ }$ $K|_0$

$P_{BG} \leftrightarrow \varphi_0 Q_{pt} \otimes Q_c$

Find global square root, so Q_c is trivial

$P_{BG} \to \varphi_0 Q_{pt} = \mathbb{Q}$

So for Loc_G^M we also get

P is $\mathbb{Q}$

$T^3 \quad GL_1 \quad \pi_1(T_3) = \mathbb{Z}^3$

$Loc_{T^3}(GL_1) = [\mathbb{C}^{\times 3}/\mathbb{C}^\times]$

Orientations correspond to functions of $\mathcal{O}_{\mathbb{C}^{\times 3}}$

orientation $\varphi = 1 \rightsquigarrow$ get constant sheaf $\xrightarrow{\text{Take cohomology}}$ $H_{\mathbb{C}^*}$

$\varphi = \frac{1}{xyz} \rightsquigarrow$ get a non-trivial local system L $\rightsquigarrow$ $\neq$ $H_{\mathbb{C}^*}(\mathbb{C}^{\times 3}, L)$

$$M = \mathbb{R}^3$$

$$R\mathrm{Loc}_G(\mathbb{R}^3) = R\mathrm{Loc}_G(pt) = BG$$

BG cannot have -1 shifted symplectic structure