BPS Lie algebras and Nakajima quiver varieties

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(-1) Ben explained to us that one can produce a Lie algebra $\mathfrak{n}^+_{\Pi_Q}$ out of the geometry of moduli spaces and stacks associated to Q.

He stated the theorem:

Thm (D-SM): $\mathfrak{n}^+_{\Pi_Q} \cong \mathrm{Free}_{\mathrm{Lie}}\left(\bigoplus_{d \in \Phi^+_{\Pi_Q}} \underbrace{\mathrm{IH}(\mathcal{M}_{\Pi_Q, d})}_{} \right) \Big/$ Serre relations

intersection cohomology

$\subset \mathfrak{g}_{\Pi_Q} = \mathfrak{n}^-_{\Pi_Q} \oplus \mathfrak{h} \oplus \mathfrak{n}^+_{\Pi_Q}$

preprojective algebra

roots $\subset \mathbb{N}^{Q_0}$

moduli space of ssimple, d-dim Π_Q-reps.

Something of the form $\mathrm{ad}(a)^M(b) = 0$

This is one key in the proof of Okounkov's conjecture (Botta-Davison), see Ben's talks.

Today: Explain this in as much detail as time allows, and also how this Lie algebra acts on the cohomology of quiver varieties.

⓪ Preprojective algebra and moduli stacks/spaces

$Q = (Q_0, Q_1)$ quiver

$\underbrace{Q_0}_{\text{vertices}}$, $\underbrace{Q_1}_{\text{arrows}}$

$$\begin{tikzcd} 1 \arrow[loop above, "\beta"] \arrow[r, "\alpha"] & 2 \end{tikzcd}$$

loops, multiple edges, 2-cycles allowed: no restrictions on Q.

source and target maps $s, t : Q_1 \to Q_0$

A representation of Q is the data of

- a vector space V_i for any vertex $i \in Q_0$
- a linear map $V_{s(\alpha)} \to V_{t(\alpha)}$ for any arrow $\alpha \in Q_1$.

dimension vector $d = (d_i = \dim V_i)_{i \in Q_0}$.

Free path algebra of Q: $\mathbb{C}Q$. Representations of Q = Rep. of $\mathbb{C}Q$.

Representation space of d-dimensional Q-representations:

$$X_{Q,d} = \prod_{\alpha \in Q_1} \mathrm{Hom}\left(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}}\right)$$

$$\circlearrowleft$$

$$GL_d := \prod_{i \in Q_0} GL_{d_i}.$$

$$\left\{ \begin{matrix} \text{isoclasses of } d\text{-dimensional} \\ Q\text{-representations} \end{matrix} \right\} \longleftrightarrow \begin{matrix} GL_d\text{-orbits in} \\ X_{Q,d} \end{matrix}$$

Stack of representations of Q

$$\begin{tikzcd}
\mathfrak{M}_Q = \bigsqcup_{d \in \mathbb{N}^{Q_0}} \mathfrak{M}_{Q,d} \arrow[d, "JH"'] \\
\mathcal{M}_Q = \bigsqcup_{d \in \mathbb{N}^{Q_0}} \mathcal{M}_{Q,d}
\end{tikzcd}$$

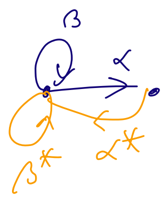
$$\mathfrak{M}_{Q,d} = X_{Q,d}/GL_d$$

$$\mathcal{M}_{Q,d} = X_{Q,d}/\!\!/GL_d = \mathrm{Spec}\left(\mathbb{C}[X_{Q,d}]^{GL_d}\right)$$

Preprojective algebra:

$Q \rightsquigarrow \overline{Q} = (Q_0, \overline{Q_1})$ double quiver

$$\overline{Q_1} = Q_1 \sqcup Q_1^{op}$$



$\mathbb{C}\overline{Q}$ free path algebra

$$\rho = \sum_{\alpha \in Q_1} [\alpha, \alpha^*] = [\alpha, \alpha^*] + [\beta, \beta^*]$$ preprojective relation

$$\Pi_Q = \mathbb{C}\overline{Q} / \langle\langle \rho \rangle\rangle$$ 2-sided ideal.

If $Q =$ , $\Pi_Q = \mathbb{C}[x, y]$, fin. dim Π_Q-representations are finite length sheaves on $\mathbb{A}^2$.

$$X_{\bar{Q},d} \underset{\text{trace}}{\cong} T^*X_{Q,d} \xrightarrow{\mu_d} \mathfrak{gl}_d^* \overset{\text{trace}}{\cong} \mathfrak{gl}_d$$

$$(x_\alpha)_{\alpha\in\bar{Q}_1} \longmapsto \sum_{\alpha\in Q_1} [x_\alpha, x_{\alpha^*}]$$

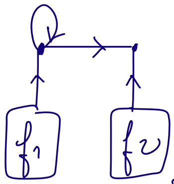
$$\begin{tikzcd} \mathfrak{M}_{\Pi_Q,d} := \mu_d^{-1}(0)/GL_d \arrow[d, "JH"'] \\ \mathcal{M}_{\Pi_Q,d} := \mu_d^{-1}(0)/\!/GL_d . \end{tikzcd}$$

good moduli space.

What makes the geometry of this stack particularly nice is that it is the classical truncation of a 0-shifted symplectic stack.

① Nakajima quiver varieties (1990s)

Nakajima quiver varieties were initially defined using the framed quiver $Q_f' =$



$f \in \mathbb{N}^{Q_0}$ framing vector.

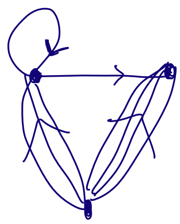
fixed dimension

(but equivalent)

We use an alternative description, allowing us to see them as moduli spaces of preprojective algebras.

$$Q_f = \left(Q_0 \cup \{\infty\} \, , \, Q_1 \cup \left\{ \alpha_{i,\ell} : \begin{array}{ll} i \in Q_0 & s(\alpha_{i,\ell}) = \infty \\ 1 \leq \ell \leq f_i \, , & t(\alpha_{i,\ell}) = i \end{array} \right\} \right)$$

$=$



$f = (3,4)$.

We look at $(d,1)$-representations of Π_{Q_f}, $d \in \mathbb{N}^{Q_0}$.

There exists a $GL_{(d,1)}$-linearization of the trivial line bundle on $X_{\bar{Q},d}$ such that a representation of $\bar{Q}$ is semi-stable iff it is generated by the 1-dim vector space V_∞.

Using King's stability conditions instead, we can take

$$\theta = (-1, \dots, -1, +|d|).$$

In this situation, stable = semistable.

Quiver variety : $N_Q(f,d) := \mu_{(d,1)}^{-1}(0)^{st} / GL_d$

free quotient.

= moduli space of (semi)stable $(d,1)$-reps of Π_{Q_f}.

Since stable = semistable, this is a smooth quasiprojective variety.

$$N_{Q,f} := \coprod_{d \in \mathbb{N}^{Q_0}} N_Q(f,d) \qquad \text{quiver variety.}$$

$$\mathbb{N}_{Q,f} := H^*(N_{Q,f}) \qquad \bigoplus_{d \in \mathbb{N}^{Q_0}} H^*(N_Q(f,d), \mathbb{Q}^{vir}).$$

(shifted) singular cohomology.

(2) Generalized Kac-Moody Lie algebras

$M = \mathbb{N}^{Q_0}$ monoid

$\mathfrak{h} = \mathbb{Q}^{Q_0}$ vector space

$\mathfrak{h} \times \mathfrak{h} \to \mathbb{Q}$ bilinear form

$\Phi^+ \subset M$ set of positive roots ; $i \in \Phi^+ \rightsquigarrow h_i \in \mathfrak{h}$.

verifying $\begin{cases} (h_i, h_{i'}) \leq 0 & \forall i \neq i' \in \Phi^+ \\ (h_i, h_i) \in 2\mathbb{Z}_{\leq 1} & \forall i \in \Phi^+ \end{cases}$

$\mathcal{G} = \bigoplus_{i \in \Phi^+} \mathcal{G}_i$ space of positive Chevalley generators.

$\mathcal{G}_i$: $\mathbb{Z}$-graded vector space, finite-dim graded parts.

$\mathfrak{g}$ is the Lie algebra generated by $\mathcal{G} \oplus \mathfrak{h} \oplus \underbrace{\mathcal{G}^\vee}_{\text{graded dual of } \mathcal{G}}$ modulo the relations

* $[h, h'] = 0 \quad \forall h, h' \in \mathfrak{h}$
* $[h, \alpha_i^\vee] = \pm (h, h_i) \alpha_i^\vee \qquad \alpha_i^\vee \in \mathcal{G}_i^\vee$
* $[\alpha_i, \alpha_j^\vee] = \delta_{ij}\, \alpha_j^\vee(\alpha_i)\, h_i$

Serre relations
* $[\alpha_i^\vee, -]^{1-(h_i, h_j)} = 0$ if $(h_i, h_j) = 0$ or $(h_i, h_i) = 2$

Trichotomy of roots

Roots $i \in \Phi^+$ come in three kinds:

* real : $(h_i, h_i) = 2$
* isotropic : $(h_i, h_i) = 0$
* hyperbolic : $(h_i, h_i) < 0$.

New in GKMs

triangular decomposition:

$$\mathfrak{g} = \mathfrak{n}_{\mathfrak{g}}^{+} \oplus \mathfrak{h} \oplus \mathfrak{n}_{\mathfrak{g}}^{-}$$

$$\mathfrak{n}_{\mathfrak{g}}^{+} = \langle \mathfrak{g} \rangle / \text{Serre relations} \qquad \mathfrak{n}_{\mathfrak{g}}^{-} = \langle \mathfrak{g}^{\vee} \rangle / \text{Serre relations}$$

<u>Examples</u>: $Q = (Q_0, Q_1)$ quiver

$$\mathfrak{h} = \mathbb{Q}^{Q_0}$$

$$(-,-) : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{Q}$$

$$(d, e) \longmapsto 2 \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} \left(d_{s(\alpha)} e_{t(\alpha)} - e_{s(\alpha)} d_{t(\alpha)} \right)$$

symmetrized Euler form.

① Q has no loops

$\Phi^+ = Q_0 \subset \mathbb{N}^{Q_0}$ coordinate vectors

$\mathcal{G}_i = \mathbb{Q}$ $\quad i \in Q_0$

$\rightarrow \mathfrak{g}_\mathcal{G}$ is the Kac-Moody algebra classically associated with Q

$\mathfrak{g}_\mathcal{G} = \mathfrak{g}_Q$

<u>subexample</u>: Q Dynkin ADE $\rightsquigarrow$ associated semisimple simply-laced Lie algebra.

② $Q = $ (one vertex with one loop) $\quad \Phi^+ = \mathbb{Z}_{\geq 1} \subset \mathbb{Q} := \mathfrak{h}$

$\mathcal{G}_n = \mathbb{Q}[-2] \quad n \geq 1 \qquad$ 1-dim vector space in degree 2

$\mathfrak{heis} := \mathfrak{g}_\mathcal{G}$ is ∞-dimensional Heisenberg Lie algebra acting on the cohomology of Hilbert schemes of points on $\mathbb{C}^2$.

<u>Involution</u>: Choose a basis of $\mathcal{G}$ $\quad (b)$; $(b^\vee)$ dual basis

$$\omega : \mathcal{G} \longrightarrow \mathcal{G}^\vee \qquad \mathcal{G}^\vee \to \mathcal{G} \qquad \mathfrak{h} \to \mathfrak{h}$$
$$b \longmapsto -b^\vee \qquad b^\vee \mapsto -b \qquad h \mapsto -h$$

induces $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ automorphism of $\mathfrak{g}$.

<u>Invariant scalar product</u>:

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{Q} \qquad \begin{cases} (\alpha_i, \alpha_i^\vee) = \alpha_i^\vee(\alpha_i) \\ \mathfrak{g}_i \perp \mathfrak{g}_j \text{ if } i+j \neq 0 \\ (x, [y,z]) = ([x,y], z) \end{cases}$$

Lowest weight representations of GKMs

$\xi : \mathfrak{h} \to \mathbb{Q}$ linear form gives a 1-dim representation

$$U(\mathfrak{h}) \to \mathbb{Q}$$

and $\underbrace{U(\mathfrak{n}^- \oplus \mathfrak{h})}_{U(\mathfrak{b}^-)} \longrightarrow U(\mathfrak{h}) \longrightarrow \mathbb{Q}$

which can be induced to

$$M_\xi := U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} \mathbb{Q} \quad \text{rep of } \mathfrak{g}.$$

$$\cup$$

$1 \otimes 1$ highest weight vector.

$$L_\xi := M_\xi / \text{maximal proper submodule}$$

is the simple lowest weight rep of $\mathfrak{g}$ of weight ξ.

(34) Action of GKMs on the cohomology of quiver varieties following Nakajima, Nakajima/Grojnowski.

<u>Theorem</u> (Nakajima, 1990s)
There is an action of $\mathfrak{g}_Q$ on $\mathbb{N}_{Q,f}$, isomorphic to the simple lowest weight representation of $\mathfrak{g}_Q$ of weight $(f,-)$.

<u>Theorem</u> (Nakajima, Grojnowski)
There is an action of heis on $\bigoplus_{n\geq 0} H^*(\mathrm{Hilb}^n \mathbb{C}^2)$, isomorphic to the Fock representation.

These actions are constructed via correspondences.

I will explain a cohomological Hall algebra approach to these constructions. It is a generalization and unification of both these theorems.

⑤ Cohomological Hall algebras and BPS Lie algebras

Q quiver

Π_Q preprojective algebra of Q

$\mathfrak{M}_{\Pi_Q} \xrightarrow{JH} \mathcal{M}_{\Pi_Q}$ good moduli space.

Exact_{Π_Q} stack of short exact sequences of Π_Q- representations.
Can be constructed as the stack of pairs of a rep. with a sub-representation.

$$\begin{tikzcd}
& \mathrm{Exact}_{\Pi_Q} \arrow[dl, "q"'] \arrow[dr, "p \text{ proper}"] & \\
\mathfrak{M}_{\Pi_Q} \times \mathfrak{M}_{\Pi_Q} \arrow[d, "JH \times JH"'] \arrow[rr, phantom, "\circlearrowright"] & & \mathfrak{M}_{\Pi_Q} \arrow[d, "JH"] \\
\mathcal{M}_{\Pi_Q} \times \mathcal{M}_{\Pi_Q} \arrow[rr, "\oplus"'] & & \mathcal{M}_{\Pi_Q}
\end{tikzcd}$$

$p_* q^*$ gives an algebra structure on the constructible complex $\mathcal{A}_{\Pi_Q} := JH_* \mathbb{D}\mathbb{Q}^{vir}_{\mathfrak{M}_{\Pi_Q}} \in \mathcal{D}^+_c(\mathcal{M}_{\Pi_Q})$.

The monoidal structure $\boxdot$ is given by

$$\mathcal{F} \boxdot \mathcal{G} := \oplus_* (\mathcal{F} \boxtimes \mathcal{G})$$

The multiplication is a map $m : \mathcal{A} \boxdot \mathcal{A} \longrightarrow \mathcal{A}$.

Proposition (Davison): $\mathcal{A}_{\Pi_Q}$ is concentrated in perverse degrees ≥ 0.

Corollary: $\mathcal{BPS}_{\Pi_Q, \underbrace{Alg}_{\text{short version of "associative algebra"}}} := {}^{\mathfrak{p}}\mathcal{H}^0(\mathcal{A}_{\Pi_Q})$ has an induced multiplication.

The natural adjunction morphism

$$\mathcal{BPS}_{\Pi_Q, Alg} \longrightarrow \mathcal{A}_{\Pi_Q}$$

is an algebra map.

$$A_{\Pi_Q} := H^*(\mathcal{A}_{\Pi_Q}) \quad \text{actual algebra}$$

$$\cup$$

$$BPS_{\Pi_Q, Alg} := H^*(\mathcal{BPS}_{\Pi_Q, Alg}) \quad "$$

We would like a more concrete description of $BPS_{\Pi_Q, Alg}$.

Theorem (D-SM)

$$\mathrm{BPS}_{\Pi_Q} \cong U(\mathfrak{n}^+_{\Pi_Q}) \quad \text{where}$$

$\mathfrak{n}^+_{\Pi_Q}$ is the positive part of the GKM associated to the data

$M = \mathbb{N}^{Q_0}$

$(-,-) : \mathbb{Q}^{Q_0} \times \mathbb{Q}^{Q_0} \longrightarrow \mathbb{Q}$ symmetrized Euler form

$\phi^+ = \Sigma_{\Pi_Q} \cup \{ l d \ , \ d \in \Sigma_{\Pi_Q}, (d,d) = 0, \ \ l \geq 2 \}$

$$\mathfrak{g}_d = \begin{cases} \mathrm{IH}(\mathcal{M}_{\Pi_Q,d}) & \text{for } d \in \Sigma_{\Pi_Q} \\ \mathrm{IH}(\mathcal{M}_{\Pi_Q,d'}) & \text{for } d = ld' \ , \ d' \in \Sigma_{\Pi_Q}, \\ & (d',d') = 0, \ l \geq 2 . \end{cases}$$

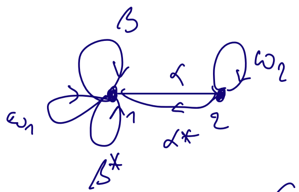
Actually, we prove a sheafified version of this theorem.

Geometric definition of $\mathfrak{n}^+_{\Pi_Q}$: quiver with potential and dimensional reduction

$Q \rightsquigarrow \overline{Q} \rightsquigarrow \tilde{Q}$ triple quiver

$(Q_0, Q_1) \quad (Q_0, \overline{Q_1}) \quad (Q_0, \tilde{Q}_1)$

$\tilde{Q}_1 = \overline{Q_1} \sqcup \{\omega_i : i \in Q_0\} \qquad s(\omega_i) = t(\omega_i) = i$



cubic potential $W = \left(\sum_{i \in Q_0} \omega_i\right)\left(\sum_{\alpha \in Q_1} [\alpha, \alpha^*]\right) \in \mathbb{C}\tilde{Q}$.

$$\begin{array}{ccc} \mathrm{Tr}W : \mathfrak{M}_{\tilde{Q}} & \longrightarrow & \mathbb{C} \\ \quad \downarrow{\scriptstyle \mathrm{JH}} & & \\ \mathcal{M}_{\tilde{Q}} & & \end{array}$$

$$\mathcal{BPS}_{\tilde{Q},W} := {}^p\mathcal{H}^1\left(\mathrm{JH}_* \phi_{\mathrm{Tr}W} \mathbb{Q}^{\mathrm{vir}}\right) \in \mathrm{Perv}(\mathcal{M}_{\tilde{Q}})$$

$$\mathcal{M}_{\Pi_Q} \times \mathbb{C} \overset{i}{\hookrightarrow} \mathcal{M}_{\tilde{Q}}$$

Prop (Davison) $\mathcal{BPS}_{\tilde{Q},W} = i_*\left(\mathcal{BPS}_{\Pi_Q,\mathrm{Lie}} \boxtimes \mathbb{Q}[1]\right)$ for some $\mathcal{BPS}_{\Pi_Q,\mathrm{Lie}} \in \mathrm{Perv}(\mathcal{M}_{\Pi_Q})$.

Proposition We have $\mathfrak{n}^+_{\Pi_Q} \cong H^*(\mathcal{BPS}_{\Pi_Q,\mathrm{Lie}})$ in a canonical way.

Invariance property of the BPS sheaf

Let ξ be a stability condition for Q.

$$\begin{tikzcd} \mathcal{M}^{\xi\text{-sst}}_{\Pi_Q,d} \arrow[d, "p"'] \\ \mathcal{M}_{\Pi_Q,d} \end{tikzcd}$$

Proposition (Toda): $p_* \mathcal{BPS}^{\xi\text{-sst}}_{\Pi_Q,d} = \mathcal{BPS}_{\Pi_Q,d}$

If stable = semistable, $\mathcal{BPS}^{\xi\text{-sst}}_{\Pi_Q,d} \cong \mathbb{Q}_{\mathcal{M}^{\xi\text{-sst}}_{\Pi_Q,d}}[\dim \mathcal{M}^{\xi\text{-sst}}_{\Pi_Q,d}]$.

Consequence: Studying cohomology of smooth quiver varieties amounts to studying BPS sheaf for singular ones.

6) Action on the cohomology of quiver varieties

Theorem (D-SM) $\mathfrak{g}_{\Pi_Q}$ acts on $\mathbb{N}_{Q,f}$ and we have the decomposition

$$\mathbb{N}_{Q,f} = \bigoplus_{(d,1)\in\Sigma_{\Pi_{Q_f}}} \underbrace{\mathrm{IH}(\mathcal{M}_{\Pi_{Q_f},(d,1)})}_{\text{multiplicity}} \otimes L((d,1),-)_{Q_f}$$

Proof: With the prelimenary work, it is almost formal.

Idea: $\mathfrak{n}^+_{\Pi_Q} \subset \mathfrak{n}^+_{\Pi_{Q_f}}$

$$\mathfrak{n}^+_{\Pi_{Q_f}} \supset \bigoplus_{d\in\mathbb{N}^{Q_0}} \mathfrak{n}^+_{\Pi_{Q_f}}[d,1] \cong \mathbb{N}_{Q,f}.$$

This inclusion induces the action of $\mathfrak{n}^+_{\Pi_Q}$ on $\mathbb{N}_{Q,f}$.

* This action is semisimple: $\mathbb{N}_{Q,f}$ has a $\mathfrak{n}^+_{\Pi_Q}$-invariant scalar product.
* $\bigoplus_{(d,1)\in\Sigma_{\Pi_{Q_f}}} \mathrm{IH}(\mathcal{M}_{\Pi_{Q_f},(d,1)}) \subset \mathbb{N}_{Q,f}$ is a space of lowest weight vectors.
* By the PBW theorem, ~~these are~~ this subspace generates $\mathbb{N}_{Q,f}$ as a $\mathfrak{n}^+_{\Pi_Q}$-representation
* $\mathrm{IH}(\mathcal{M}_{\Pi_{Q_f},(d,1)})$ is of lowest weight $((d,1),-)_{Q_f}: \mathbb{Q}^{Q_0} \to \mathbb{Q}$.

We retrieve Nakajima's theorem by considering H^0 only in the previous theorem.

Prop (Davison) $H^0(n^+_{\Pi_Q}) \simeq n^+_{Q^{re}}$ is the positive part of the KM algebra associated to Q^{re}, the real subquiver of Q: it is the full subquiver containing loop-free vertices of Q.

More consequences of the main theorem

e.g positivity of Kac polynomials of quivers