Cohomological integrality isomorphisms

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Cohomological integrality isomorphisms

produces some decompositions of polynomial rings

①

$$P_{d_0} = \mathbb{Q}[x]_{\deg < d} \qquad d \geq 0$$

$$P_{d_1} = \mathbb{Q}.$$

Integrality isomorphism

$$P_{d_0} \oplus \left(P_{d_1} \otimes \mathbb{Q}[x]\right) \longrightarrow \mathbb{Q}[x] \qquad C \in \mathbb{Q}\setminus\{0\}$$

$$(f, g) \longmapsto f + Cx^d \cdot g.$$

clearly an isomorphism

②

$$P_{d_0} \oplus \left(P_{d_1} \otimes \mathbb{Q}[x_1]\right) \oplus \left(P_{d_2} \otimes \mathbb{Q}[x_1, x_2]\right)^{sgn} \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$(f, h, k) \mapsto f + \frac{x_1^g h(x_1, x_2) - x_2^g h(x_2, x_1)}{x_1 - x_2} + 2(x_1 x_2)^g \frac{k(x_1, x_2)}{x_1 - x_2}.$$

$$P_{d_0} = \bigoplus_{j=0}^{g-2} \mathbb{Q}(x_1+x_2)^j \subset H^*(V/G) \cong \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$P_{d_1} = \bigoplus_{j=0}^{g-1} \mathbb{Q}x_2^j \subset H^*(V^{d_1}/G^{d_1}) \cong \mathbb{Q}[x_1, x_2]$$

$$\mathcal{P}_{d_2} = \mathbb{Q} \subset H^*(V^{d_2}/G^{d_2}) \cong \mathbb{Q}[x_1, x_2]$$

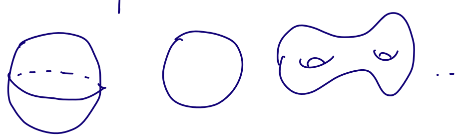
$$\mathcal{P}_{d_3} = \{0\} \subset \mathbb{Q}[x_1+x_2,\ x_1x_2].$$

1. Cohomological integrality

Cohomology

X topological space $\rightsquigarrow$ $H^*(X)$ $\mathbb{Z}$-graded vector space

~1930



X differential manifold $\rightsquigarrow$ $H^i(X) = 0$ for $i<0$ or $i > \dim X$

$\dim H^0(X) = \#$ connected components of X

$H^1(X)$ related to embeddings of circles $\bigcirc$ in X

$H^*(X)$ enriches / categorifies the Euler characteristic $\chi(X)$.

$$\chi(X) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(X)$$

Intersection cohomology

Refinement of singular cohomology. $\mathbb{Z}$-graded vector space.

X algebraic variety $\rightsquigarrow$ $IH^*(X)$ intersection cohomology

Goresky-MacPherson

~1980

X smooth $\Rightarrow$ $H^*(X) \cong IH^*(X)$

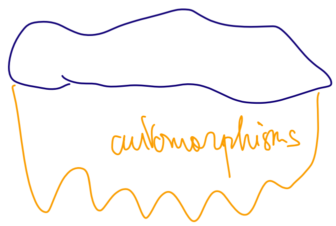
X singular $\rightsquigarrow$ $IH^*(X)$ records some information about singularities

Stacks

class of geometric objects, quite close to algebraic varieties but more flexible.

intuitively: $\sim$ orbifold

$\sim$ geometric object + each point comes with an algebraic group (automorphism group)



examples: X algebraic variety, G group $[X/G]$ stack

$X = pt$ $\quad [pt/G]$

things on $[X/G]$ are things on X with symmetries given by G.

Rough statement of cohomological integrality

Theorem: (H, 2024)

Let $\mathfrak{M}$ be a stack satisfying some assumptions. Then, there exists a finite dimensional subspace $P_{\mathfrak{M}} \subset H^*(\mathfrak{M})$ which generates (in the sense of parabolic induction).

Stacks of interest: $[V/G]$ where V is a representation of the red group G.

2- Symmetric representations of reductive groups

$G = GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $(\mathbb{C}^*)^N$, $Sp_{2n}(\mathbb{C})$, ...

More generally, G: reductive group (unipotent radical is trivial)

$\underset{\text{char } 0}{=}$ linearly reductive

(finite-dimensional representations are semisimple)

<u>non-example</u>: $G = \mathbb{G}_a$ additive group

acts on $V = \mathbb{C}^2$ via $\mathbb{G}_a \hookrightarrow \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$.

V is non-trivial extension of $\mathbb{C}$ by $\mathbb{C}$.

* $T \subset G$ maximal torus. $T \simeq (\mathbb{C}^*)^{\mathrm{rank}(G)}$

e.g. diag $\simeq (\mathbb{C}^*)^n \subset GL_n(\mathbb{C})$.

* representation: $G \longrightarrow GL(V)$, V $\mathbb{C}$ vector space, finite-dimensional.

$GL_2(\mathbb{C})$, $SL_2(\mathbb{C}) \curvearrowright \mathbb{C}^2$.

characters: $X^*(T) = \{\alpha : T \to \mathbb{G}_m\} \cong \mathbb{Z}^{\mathrm{rk}\, G}$

cocharacters $X_*(T) = \{\lambda : \mathbb{G}_m \to T\} \cong \mathbb{Z}^{\mathrm{rk}\, G}$

Pairing $\alpha \circ \lambda : \mathbb{G}_m \longrightarrow \mathbb{G}_m$
$$z \longmapsto z^{\langle \lambda, \alpha \rangle}$$
$$\langle -,- \rangle : X_*(T) \times X^*(T) \to \mathbb{Z}.$$

Weights $T \curvearrowright V$ diagonalizable:
$$V = \bigoplus_{\alpha \in X^*(T)} V_\alpha$$
$$V_\alpha = \{v \in V \mid t \cdot v = \alpha(t) \cdot v \quad \forall t \in T\}$$
$$\mathcal{W}(V) = \{\alpha \in X^*(T) \mid V_\alpha \neq 0\} \text{ weights of } V.$$

In particular, $\mathcal{W}(\mathfrak{g})$ weights of $\mathfrak{g} = \mathrm{Lie}(G)$

ex. $GL_2(\mathbb{C}) \curvearrowright \mathbb{C}^2 = \underset{(1,0)}{\mathbb{C}e_1} \oplus \underset{(0,1)}{\mathbb{C}e_2}$
$$\cup$$
$$(\mathbb{C}^*)^2$$
$$(t_1, t_2)e_1 = t_1 e_1$$
$$(t_1, t_2)e_2 = t_2 e_2$$

V symmetric: $\dim V_\alpha = \dim V_{-\alpha} \quad \forall \alpha \in X^*(T)$

$\Leftrightarrow \; V \cong V^*$ (a representation is determined by its character)

sort of weakening of symplecticity, appears sometimes when def Coulomb branches.

ex: $\cdot\; T^*V = V \oplus V^*$, V rep of G

$\cdot$ any V rep of $SL_2(\mathbb{C})$

$\cdot\; \mathfrak{g}$ adjoint of G

$O(2n+1)$ $\quad Sp(2n)$

$\cdot$ any representations in type B_n, C_n, E_7, E_8, F_4

Weyl group $\quad W = N_G(T)/T$

$$N_G(T) = \{g \in G \mid gTg^{-1} = T\}$$

$$W_{GL_n} \cong W_{SL_n} \cong \mathfrak{S}_n \text{ symmetric group.}$$

In general: W is a Coxeter group.

T torus: $W_T = \{e\}$

$W \curvearrowright$ weights of $V = \mathcal{W}(V)$.

Cohomological integrality

$H^*_G(V)$ equivariant cohomology

V v.space $\Rightarrow$ contractible

$$H^*_G(V) \cong H^*_G(pt) \cong H^*(BG)$$

E_G contractible space with free G action.

$BG = EG/G$.

ex: $H^*_{\mathbb{C}^*}(pt)$

$G = \mathbb{C}^* \curvearrowright \mathbb{C}^N \setminus \{0\}$ free

$$\mathbb{C}^\infty \setminus \{0\} = \bigcup_N \mathbb{C}^N \setminus \{0\}$$

$$\mathbb{P}^\infty = \mathbb{C}^\infty \setminus \{0\} / \mathbb{C}^*$$

$$H^*(\mathbb{P}^N) \cong \mathbb{Q}[x]/(x^{N+1}) \qquad \deg(x) = 2$$

$$H^*(\mathbb{P}^\infty) = H^*_{\mathbb{C}^*}(pt) \cong \mathbb{Q}[x]$$

$G = T = (\mathbb{C}^*)^n$ $\qquad H^*_T(pt) \cong \mathbb{Q}[x_1, \dots, x_n]$

G general $\qquad H^*_G(pt) \cong H^*_T(pt)^W \qquad T \subset G$ max torus.

$\rightsquigarrow H^*_G(pt) \cong \mathbb{Q}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ symmetric polynomials

$G = GL_2(\mathbb{C}) \qquad \mathbb{Q}[x_1 + x_2, x_1 x_2]$

In general $H^*_G(pt)$ is a polynomial algebra

in particular, $\dim_{\mathbb{Q}} H^*_G(pt) = +\infty$.

Cohomological integrality

Extract a finite-dimensional subspace

$$P_0 \subset H^*(V/G)$$

that generates (in the sense of parabolic induction)

P_0 = " cuspidal cohomology " of V/G.

$\hookrightarrow$ analogy with $\begin{cases} \text{character sheaves (rep of fin. grps of Lie type)} \\ \text{Hecke eigensheaves (Langlands)} \end{cases}$

③ Operations

Parabolic induction

V representation of G

$\lambda : \mathbb{G}_m \to T$ cocharacter

$G^\lambda = \{ g \in G \mid \lambda(t) g \lambda(t)^{-1} = g \quad \forall t \in \mathbb{C}^* \}$

$\subset G$ Levi subgroup

Note G^λ reductive

$T \subset G^\lambda$.

$V^\lambda = \{ v \in V \mid \lambda(t) v \lambda(t)^{-1} = v \quad \forall t \in \mathbb{C}^* \}$

$\subset V$ subspace

$G^{\lambda \geq 0} = \{ g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists} \}$

$\subset G$ parabolic subgroup

$V^{\lambda \geq 0} = \{ v \in V \mid \lim_{t \to 0} \lambda(t) \cdot v \text{ exists} \}$

$\subset V$

subspace

$G = GL_n \qquad V = T^* \mathbb{C}^n$

$\mathbb{G}_m \longrightarrow GL_n$

$t \longmapsto \begin{pmatrix} t^2 & & 0 \\ & t & \\ 0 & & 1 \end{pmatrix}$

$\begin{pmatrix} \boxed{*} & & 0 \\ & \boxed{*} & \\ 0 & & \boxed{*} \end{pmatrix}$

$V^\lambda = T^* \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$

$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$V^{\lambda \geq 0} = \mathbb{C}^n \oplus \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$

Induction diagram

$$\begin{tikzcd}
 & V^{\lambda \geq 0}/G^{\lambda \geq 0} \arrow[dl, "q_\lambda"', "\text{smooth}"] \arrow[dr, "p_\lambda", "\text{proper}"'] & \\
V^\lambda / G^\lambda & & V/G
\end{tikzcd}$$

$$\mathrm{Ind}_\lambda := p_{\lambda *} q_\lambda^* : H^*(V^\lambda/G^\lambda) \longrightarrow H^*(V/G)$$

parabolic induction

$$\mathrm{Ind}_\lambda : \mathbb{Q}[x_1, \dots, x_r]^{W^\lambda} \longrightarrow \mathbb{Q}[x_1, \dots, x_r]^W.$$

$\exists$ translation of coh degree making Ind_λ graded.

Explicit formula:

$$k_\lambda := \frac{\prod_{\substack{\alpha \in \mathcal{W}(V) \\ \langle \lambda, \alpha \rangle > 0}} \alpha^{\dim V_\alpha}}{\prod_{\substack{\alpha \in \mathcal{W}(\mathfrak{g}) \\ \langle \lambda, \alpha \rangle > 0}} \alpha^{\dim \mathfrak{g}_\alpha}} \in \mathrm{Frac}\left(H^*_T(pt)\right)$$

$\alpha \in X^*(T)$ may be seen as an element of $H^*_T(pt) \cong \mathrm{Sym}(\mathfrak{t}^*)$

$\alpha : T \to \mathbb{G}_m \quad d\alpha(1) : \mathfrak{t} \to \mathbb{C}$.
$\in \mathfrak{t}^*$

$$\mathrm{Ind}_\lambda(f) = \frac{1}{|W^\lambda|} \sum_{w \in W} w \cdot (f k_\lambda).$$

Proof: Calculation after localization and computation of Euler class, using Borel-Weil-Bott thm.

3. Main theorem

Cohomological integrality theorem

$$\lambda \sim \mu \iff \begin{cases} G^\lambda = G^\mu \\ V^\lambda = V^\mu \end{cases}$$

$$\rightsquigarrow \quad \mathcal{P}_V = X_*(T)/\sim \quad \text{finite set}$$

$$\circlearrowleft W$$

$$G_\lambda = \ker\left(G^\lambda \longrightarrow GL(V^\lambda)\right) \cap Z(G^\lambda) \subset G \quad \text{normal subgroup.}$$

$$W_\lambda = \{ w \in W \mid w\cdot\lambda \sim \lambda \} \subset W \quad \text{subgroup}$$

$$\varepsilon_{V,\lambda} : W_\lambda \longrightarrow \{\pm 1\} \quad \text{such that}$$

$$k_{w\cdot\lambda} = \varepsilon_{V,\lambda}(w)\, k_\lambda \qquad \forall w \in W_\lambda .$$

Thm (H, 2024) Let V be a symmetric (= self-dual) representation of G. For $\lambda \in X_*(T)$, $\exists\ P_\lambda \subset H^*_{G^\lambda}(V^\lambda)$ finite-dimensional and graded, stable under the W_λ-action, s.t.

isotypic component

$$\bigoplus \left(P_\lambda \otimes H^*(pt/G_\lambda)\right)^{\varepsilon_{V,\lambda}} \longrightarrow H^*_G(V)$$

$\tilde{\lambda} \in P_V / W$

is a graded isomorphism + P_0 determined by the existence of such an isomorphism.

Def $p_{\lambda,i} = \dim P_\lambda^i \in \mathbb{N}$ "refined DT invariants of (G,V)".

new enumerative invariants we seek to understand and interpret geometrically.

① $\mathbb{C}^* \curvearrowright V = \bigoplus_{k\in\mathbb{Z}} V_k$ Examples

For simplicity, we assume $V_0 = 0$.

$$\lambda_0 : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$t \longmapsto 1$$

$$\lambda_1 : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$t \longmapsto t$$

$\mathcal{P}_V = \{[\bar{\lambda}_0, \bar{\lambda}_1]\}$; no Weyl group

$V^{\lambda_0} = V$, $G^{\lambda_0} = G$, $G_{\lambda_0} = \{1\}$, $k_{\lambda_0} = 1$

$V^{\lambda_1} = pt$, $G^{\lambda_1} = G$, $G_{\lambda_1} = G$, $k_{\lambda_1} = \prod_{k>0} (kx)^{\dim V_k} \in \mathbb{Q}[x]$.

$$\mathrm{Ind}_{\lambda_1,\lambda_0} : \mathbb{Q}[x] \longrightarrow \mathbb{Q}[x]$$
$$f(x) \longmapsto k_{\lambda_1} \cdot f(x)$$
$$k_{\lambda_1} = C_V \cdot x^{\sum_{k>0} \dim V_k}.$$

$$P_{\lambda_0} = \mathbb{Q}[x]_{\deg < \sum_{k>0} \dim V_k}$$

$$P_{\lambda_1} = \mathbb{Q}.$$

Integrality isomorphism

$$
\begin{aligned}
P_{d_0} \oplus \left(P_{d_1} \otimes \mathbb{Q}[x]\right) &\longrightarrow \mathbb{Q}[x] \\
(f, g) &\longmapsto f + k_{d_1} \cdot g .
\end{aligned}
$$

clearly an isomorphism

$$\textcircled{2}\quad \overset{G}{\overset{\shortparallel}{GL_2(\mathbb{C})}} \curvearrowright \overset{V}{\overset{\shortparallel}{(T^*\mathbb{C}^2)^g}} \qquad g \geq 0 \qquad\qquad T = (\mathbb{C}^*)^2 \subset GL_2(\mathbb{C})$$

$$\lambda_0 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto 1$$

$$\lambda_1 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto (t,1)$$

$$\lambda_2 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto (t,t^2)$$

$$\lambda_3 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto (t,t)$$

- $V^{\lambda_0} = V$, $G^{\lambda_0} = G$, $G_{\lambda_0} = \{1\}$, $W_{\lambda_0} = W$, $k_{\lambda_0} = 1$,
  $\varepsilon_{V,\lambda_0} = \mathrm{triv}$

- $V^{\lambda_1} = (T^*(0 \oplus \mathbb{C}))^g$, $G^{\lambda_1} = T$, $G_{\lambda_1} = \mathbb{C}^* \times \{1\}$, $W_{\lambda_1} = \{1\}$,
  $\varepsilon_{V,\lambda_1} = \mathrm{triv}$ $\qquad k_{\lambda_1} = \dfrac{x_1{}^g}{x_1 - x_2}$

- $V^{\lambda_2} = \{0\}$, $G^{\lambda_2} = T$, $G_{\lambda_2} = T$, $W_{\lambda_2} = W$
  $k_{\lambda_2} = \dfrac{(x_1x_2)^g}{x_1 - x_2}$ $\qquad \varepsilon_{V,\lambda_2} = \mathrm{sgn}$

• $V^{d_3} = \{0\}$, $G^{d_3} = G$, $G_{d_3} = G$, $W_{d_3} = W$,

$k_{d_3} = (x_1 x_2)^g$, $\varepsilon_{V, d_3} = \mathrm{sgn}$.

Some calculations:

$$P_{d_0} = \bigoplus_{j=0}^{g-2} \mathbb{Q}(x_1+x_2)^j \subset H^*(V/G) \cong \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$P_{d_1} = \bigoplus_{j=0}^{g-1} \mathbb{Q}x_2^j \subset H^*(V^{d_1}/G^{d_1}) \cong \mathbb{Q}[x_1, x_2]$$

$$P_{d_2} = \mathbb{Q} \subset H^*(V^{d_2}/G^{d_2}) \cong \mathbb{Q}[x_1, x_2]$$

$$P_{d_3} = \{0\} \subset \mathbb{Q}[x_1+x_2, x_1x_2].$$

$$\mathrm{Ind}_{d_1} : \mathbb{Q}[x_1, x_2] \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$
$$f(x_1, x_2) \longmapsto \frac{x_1^g f(x_1,x_2) - x_2^g f(x_2, x_1)}{x_1 - x_2}$$

$$\mathrm{Ind}_{d_2, d_3} : \mathbb{Q}[x_1, x_2] \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$
$$f(x_1, x_2) \longmapsto \frac{f(x_1,x_2) - f(x_2, x_1)}{x_1 - x_2}$$

surjective $\Rightarrow P_{d_3} = \{0\}$.

Integrality isomorphism

$$P_{\lambda_0} \oplus \left(P_{\lambda_1} \otimes \mathbb{Q}[x_1]\right) \oplus \left(P_{\lambda_2} \otimes \mathbb{Q}[x_1, x_2]\right)^{sgn} \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$(f, h, k) \mapsto f + \frac{x_1^g h(x_1, x_2) - x_2^g h(x_2, x_1)}{x_1 - x_2} + 2(x_1 x_2)^g \frac{k(x_1, x_2)}{x_1 - x_2}.$$

<u>exercise</u>: Check by hand this is an iso.